THE MATHEMATICAL WORK OF SAM GITLER, 1960-2003

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For the past 43 years, Sam Gitler has made important contributions to algebraic topology. Even now at age 70, Sam continues to work actively. Most of his work to date can be divided into the following six areas.

(1) Immersions of projective spaces
(2) Stable homotopy types
(3) The Brown-Gitler spectrum
(4) Configuration spaces
(5) Projective Stiefel manifolds
(6) Interesting algebraic topology text

1. Immersions of projective spaces

Sam received his PhD under Steenrod at Princeton in 1960 and spent much of the 1960's lecturing at various universities in the US and UK, but he spent enough time in Mexico to obtain with José Adem very strong results on the immersion problem for real projective spaces. The problem here is to determine for each n the smallest Euclidean space $\mathbf{R}^{d(n)}$ in which RP^n can be immersed. The related embedding problem is more easily visualized, and Sam and José also obtained results on the embedding problem for real projective spaces, but the immersion problem lends itself more readily to homotopy theory, and so it gets more attention by homotopy theorists. An immersion is a differentiable map which sends tangent spaces injectively, but need not be globally injective. The usual picture of a Klein bottle ([43]) illustrates an immersion of a 2-manifold K in $\mathbf{R}^3$, but K cannot be embedded in $\mathbf{R}^3$.

Many families of immersion and of nonimmersion results are known ([24]), but to be optimal it must be that RP^n is known to immerse in $\mathbf{R}^{d(n)}$ and to not immerse in $\mathbf{R}^{d(n)-1}$. The only optimal results that are known are, ordered chronologically:

- $n = 2^i$, $d(n) = 2^{i+1} - 1$

- nonimmersion: Milnor 1957, characteristic classes. ([52])
- immersion: Whitney 1944, differential topology. ([58])
- $n = 2^i + 1$, $d(n) = 2^{i+1} - 1$
  - nonimmersion: Milnor ([52]).
  - immersion: Sanderson 1964, obstruction theory. ([56])
- $n = 2^i + 2^j + 1, 2, 3$, $i > j \geq 2$, $d(n) = 2^{i+1} + 2^{j+1} - 2$
  - nonimmersion: Adem-Gitler 1964, secondary cohomology operations. ([2])
  - immersion: Sanderson. ([56])
- $n = 31$, $d(n) = 53$
  - nonimmersion: James 1963, Adams operations in K-theory. ([42])
  - immersion: Adem-Gitler-Mahowald 1965, obstruction theory and secondary operations. ([3])
- $n = 2^i + 2, 3$, $d(n) = 2^{i+1}$
  - nonimmersion: Baum and Browder 1965, Sq^k in projective Stiefel manifolds. ([8])
  - immersion: Sanderson. ([56])
- $n = 2^i + 4$, $d(n) = 2^{i+1} + 2$
  - nonimmersion: Gitler 1968, Sq^k in projective Stiefel manifolds. ([35])
  - immersion: Nussbaum 1970, modified Postnikov towers (MPTs). ([54])
- $n = 63$, $d(n) = 115$
  - nonimmersion: James. ([42])
  - immersion: Davis-Mahowald 1978, MPTs. ([28])
- $n = 2^i + 2^j + 4$, $i > j \geq 3$, $d(n) = 2^{i+1} + 2^{j+1} - 1$
  - nonimmersion: Randall 1985, Smale invariant. ([55])
  - immersion: Davis 1983, MPTs. ([26])
- $n = 2^i + 2^j + 11$, $i > j \geq 4$, $d(n) = 2^{i+1} + 2^{j+1} + 12$
  - nonimmersion: Singh 2003, MPTs. ([57])
  - immersion: Adem-Gitler-Mahowald. ([3])

The first unknown question is whether RP^{24} immerses in $\mathbf{R}^{38}$.

There are two main points here. First is the rash of activity in the 1960's. The immersion problem was really a hot topic then and much progress was made, with Sam right in the thick of it. Although I wasn't around then, I would guess, knowing Sam, that he was probably **the** leader in pushing all this activity forward. There were two Adem-Gitler papers ([1], [2]) involving secondary cohomology operations as applied to immersions that were really a *tour de force*. In order to combat indeterminacy, they had to use secondary operations of several variables. Modified Postnikov towers (MPTs), a form of obstruction theory which, as the above table indicates, have been a very effective tool, both for immersions and nonimmersions, were developed by Gitler and Mahowald in the 1960's ([38]). If $i_k : BO(k) \to BO$ denotes the classifying space for k-dimensional vector bundles and their stabilization, then an n-manifold M immerses in $\mathbf{R}^{n+k}$ if and only if the map $M \to BO$ which classifies its stable normal bundle lifts to $BO(k)$. The obstructions to the lifting are related to the homotopy groups of the fiber V_k of i_k, and, if the dimension of M is less than $2k - 1$, MPTs enable you to filter the obstructions according to the Adams filtration of $\pi_*(V_k)$. Most of these papers in the 1960's were published in the *Boletin*. I still consider the *Boletin* as **the** place to publish work related to immersions and projective spaces.

My second main point is that it is only for n close to 2^i or $2^i + 2^j$ that optimal values are known. The number $\alpha(n)$, defined as the number of 1's in the binary expansion of n, occurs frequently in these results. When $\alpha(n)$ is large, the gaps between known immersions and nonimmersions can become quite large. The most comprehensive results are the immersions of Milgram ([51]) obtained using bilinear maps in 1967, and the nonimmersions that I obtained in 1984 ([23]) using Brown-Peterson cohomology, following a method introduced by Sam's PhD student Luis Astey ([6]). The results are

- If $n \geq 7$, RP^n immerses in $\mathbf{R}^{2n-\alpha(n)-\langle 0,1,1,4 \rangle}$ if $n \equiv \langle 1, 3, 5, 7 \rangle$ mod 8, and
- $RP^{2(m+\alpha(m)-1)}$ does not immerse in $\mathbf{R}^{4m-2\alpha(m)}$.

The largest gaps between known immersions and nonimmersions are roughly $5[\log_2(n)] - 22$, when n is just less than a power of 2. I believe that most of the remaining improvement will need to come on the immersion side.

In 1970, Sam gave a survey talk ([36]) on immersions and embeddings at the Madison conference which was very influential for me. I was just starting my thesis work, and seeing all the methods and results on this concrete problem brought together elegantly in one place led me to want to try to make my own contributions. Sam has continued to promote this area of mathematics, for example, by training Jesus González at Rochester in such a way that he has become a major contributor to this area ([41])

2. Stable homotopy types

The question of deciding whether the stunted projective spaces FP_m^{m+k} and FP_n^{n+k} have the same stable homotopy type is a natural one for homotopy theorists. Here F is the real, complex, or quaternionic numbers, and $FP_m^{m+k} = FP^{m+k}/FP^{m-1}$. We are asking if, when suspended enough, these spaces have the same homotopy type. Because P_m^{m+k} is the Thom space of the multiple $m\xi_k$ of the Hopf line bundle ξ_k, equivalences are obtained if $m\xi_k$ and $n\xi_k$ are J-equivalent, i.e., their sphere bundles are stably fiber homotopically equivalent, while nonequivalences can be obtained by applying Adams operations in K-theory. These results usually leave a small gap of cases which take more care to resolve.

In a 1977 paper, ([32]), Sam, together with Sam Feder, completely resolved the question when F is the complex numbers or quaternions. Here is the result for CP.

Theorem *If $k \neq 2$ or 4, CP_m^{m+k} and CP_n^{n+k} have the same stable homotopy type if and only if one of the following conditions holds:*

- $m - n \equiv 0 \mod A_k$;
- $m - n \equiv 0 \mod A_{k-1}$ and $m + n \equiv 0 \mod A_k$;
- $m - n \equiv 0 \mod A_{k-1}$ and $m + n + 2(k+1) \equiv 0 \mod A_k$.

Here A_k is the J-order of the Hopf bundle over CP^k, which was introduced by Atiyah-Todd and established by Adams-Walker. I think this Feder-Gitler theorem is a very elegant result, and the proof is very nice, using clever comparisons of cofiber sequences to obtain equivalences in the cases which were not easily resolved by the standard techniques mentioned above.

Also in 1977, Sam wrote a paper with Feder and Mahowald ([33]) in which they carried out a similar program when $F = \mathbf{R}$, i.e. for stunted real projective spaces.

This time there were a few of the intermediate cases that they could not resolve. In a 1986 paper ([29]), Mark and I used more detailed analysis of the Adams spectral sequence to obtain equivalences in those cases, thus finishing the stable homotopy type problem for all projective spaces. A nice form for our result is that stunted real projective spaces are stably equivalent if and only if their J-homology groups are isomorphic and their J-cohomology groups are isomorphic as graded abelian groups (with dimension shift).

Jesus González ([39]) obtained a similar complete result for stable homotopy type of stunted mod p lens spaces when p is an odd prime, as did my PhD student Huajian Yang for stunted mod 4 lens spaces. For other stunted mod p^r lens spaces, work has been done by Kobayashi ([44]) and by González ([40]), but there are still many unresolved cases.

3. Brown-Gitler spectra

Probably Sam's most celebrated work is his construction with Ed Brown of the Brown-Gitler spectra in a 1973 paper in *Topology* ([12]). These are 2-local spectra $B(n)$ satisfying

- $H^*(B(n); \mathbf{Z}_2) \approx A/A\{\chi(\mathrm{S}\,q^i) : i > n\}$;
- If ν is the normal bundle of an n-manifold embedded in $\mathbf{R}^{n+k}$,
  then there exists a map from the Thom space $T(\nu)$ to $\Sigma^k B([\frac{n}{2}])$
  of degree 1 on the bottom cell.

Here χ denotes the canonical antiautomorphism of the mod 2 Steenrod algebra A. It was known from earlier work (1964) of Brown and Peterson ([13]) that $A\{\chi \mathrm{S}\,q^i : i > [\frac{n}{2}]\}$ were the classes vanishing on Thom classes of normal bundles of all n-manifolds, so this spectrum is minimal admitting maps from all such $T(\nu)$.

Brown and Gitler constructed the spectrum bearing their name by realizing an algebraic resolution as a tower of spectra. According to a 1985 lecture by Ed Brown on the history of Brown-Gitler spectra, Sam's expertise in the various techniques utilized in his joint work with Adem, in particular, secondary cohomology operations, played an essential role in this work.

This work was motivated by the conjecture that every n-manifold can be immersed in $\mathbf{R}^{2n-\alpha(n)}$. Brown and Peterson had developed an approach to proving this conjecture. They proved in [14] that there is a space BO/I_n and map $g_n : BO/I_n \to BO$ such that the map $M \to BO$ classifying the stable normal bundle of an n-manifold lifts to a map $M \to BO/I_n$, and $H^*(g_n; \mathbf{Z}_2)$ is a surjection with kernel the ideal consisting of exactly those Stiefel-Whitney classes which were already known to vanish on normal bundles of n-manifolds. They hoped to prove that g_n factors through a map $BO/I_n \to BO(n - \alpha(n))$, which would prove the immersion conjecture. They proved the analogous statement is true for Thom spectra, i.e. there exists a map from the Thom spectrum MO/I_n of BO/I_n to $MO(n - \alpha(n))$. They hoped to deThomify this map.

In a paper in the *Annals of Mathematics* in 1985 ([21]), Ralph Cohen gave an argument for carrying out this program. The Brown-Gitler spectra play an important role in his argument because MO/I_n splits as a wedge of Brown-Gitler spectra, and this splitting is used in constructing various maps.

It seemed that Brown-Gitler spectra were popping up everywhere in the late 1970's and early 80's. The Segal Conjecture, which stated that the completed Burnside ring of a finite group was isomorphic to the 0th stable cohomotopy group of its classifying space, was a central question in algebraic topology at that time. The Burnside ring of G is defined by applying a Grothendieck construction to the set of all finite G-sets. The Segal Conjecture was proved by Gunnar Carlsson in a 1984 paper in the *Annals of Mathematics* ([17]). As a key step, he first proved it for elementary abelian 2-groups in a paper in *Topology* in 1983 ([16]). His proof of this involved Brown-Gitler spectra in a very central way. I was working on that conjecture, too, and I remember that Mark Mahowald had suggested to me that I should consider using Brown-Gitler spectra.

Mark was very involved in the understanding and applications of Brown-Gitler spectra. In his 1977 *Topology* paper ([49]), he used them in a very central way to establish existence of an infinite family of elements η_j in the stable homotopy groups of spheres. The map η_j is in the 2^j-stem. This was the first infinite family in a fixed Adams filtration, in this case filtration 2. Prior to this result, there was a conjecture, called the Doomsday Conjecture (see [53, p.240] for historical comments), which said

that each Adams filtration contains only finitely many elements of $\pi_*(S^0)$, and so Mahowald's result doomed the Doomsday Conjecture. His proof showed that some Brown-Gitler spectra could be realized as certain summands in the stable splitting of $\Omega^2 S^9$. This set off a flurry of activity characterizing Brown-Gitler spectra and finding them in nature ([15],[22]). Mark's method of using Brown-Gitler spectra to construct infinite families in the stable homotopy groups of spheres was mimicked by W.H. Lin and Nick Kuhn to produce other infinite families ([48], [47]).

Brown-Gitler spectra also appeared centrally in Mark's splitting of $bo \wedge bo$ ([50]), which was a major tool in our 1989 *Topology* paper ([30]) on the image of J, and in a 1981 paper of Sam, Mark, and me ([27]) on stable geometric dimension. Martin Bendersky will give a talk here discussing our recent completion of that project ([9]).

4. Configuration spaces

In several papers, Sam has studied the (co)homology of configuration spaces or their loop spaces. If X is a topological space, then the configuration space $F(X, n)$ is the subspace of X^n consisting of n-tuples of distinct points of X. In a 1991 paper ([10]), Bendersky and Gitler used simplicial methods to construct a spectral sequence converging to the cohomology groups of $F(M, n)$ when M is a manifold. This was related to earlier work of Gelfand-Fuks ([34]) and Bott-Segal ([11]) on the cohomology of the Lie algebra of tangent vector fields on a manifold. Fred Cohen and Larry Taylor ([20]) had done some related work in 1978.

Fred certainly has a long-standing interest in configuration spaces from many points of view. Being together at Rochester, it was natural for Sam to get involved with Fred on some of this. It led to two papers in 2001-2 on loop spaces of configuration spaces and braid groups ([18],[19]). A pure braid with k strands in a manifold M is k paths in M which do not collide at any t and which end where they start. When you picture them as a subspace of $M \times I$, it becomes clear why they are called braids. A pure braid with k strands in M is just a loop in $F(M, k)$.

The 2002 paper of Sam and Fred ([19]) contains numerous product decomposition theorems, up to homotopy type. Some which are easy to state are

- if $m \geq 3$, $\Omega F(\mathbf{R}^m, k) \simeq \prod_{i=1}^{k-1} \Omega(\bigvee_i S^{m-1})$, and
- if $n \geq 1$, $\Omega F(S^{2n+1}, k) \simeq \Omega S^{2n+1} \times \Omega F(\mathbf{R}^{2n+1}, k-1)$.

Their main results deal with homology. One such result, which is then generalized in various ways, is that $H_*(\Omega F(\mathbf{R}^m, k))$ is isomorphic as Hopf algebras to the universal enveloping algebra of the graded Lie algebra $\mathcal{L}_k(m-2)$. This is the largest Lie algebra for which the infinitesimal braid relations are satisfied. It has been studied by Kohno ([45],[46]), in relation to Vassiliev invariants of braids. Here $m-2$ is the grading of the generators and k is the range for the subscripts of the generators $B_{i,j}$.

5. Projective Stiefel manifolds

Sam has had a longstanding interest in projective Stiefel manifolds. The Stiefel manifold $V_{n,k}$ is the space of k-frames in $\mathbf{R}^n$, and the projective Stiefel manifold $X_{n,k}$ is its quotient obtained by identifying a frame with its negative. It admits a canonical line bundle $\xi_{n,k}$. In 1968, Sam, with David Handel, determined $H^*(X_{n,k}; \mathbf{Z}_2)$ as an algebra over the Steenrod algebra, up to some indeterminacy ([37]). A main reason for their interest was that if θ is a line bundle over a space Y, then $n\theta$ has k linearly independent sections if and only if there is a map $f : Y \to X_{n,k}$ such that $f^*(\xi_{n,k}) = \theta$. As mentioned earlier, Sam, in one of his rare solo papers ([35]), was able to use this method to obtain strong new nonimmersion results for certain RP^n.

A natural question to ask about a manifold is whether it is parallelizable; i.e., if the tangent bundle is isomorphic to a trivial bundle. In a 1986 paper in the *Boletin* ([5]), Antoniano, Gitler, Ucci, and Zvengrowski answered this question for all $X_{n,k}$ except $X_{12,8}$. Their answer was that the only parallelizable ones are $X_{n,n}$, $X_{n,n-1}$, $X_{2n,2n-2}$, $X_{4,1}$, $X_{8,k}$, and $X_{16,8}$. The positive results are obtained for various special reasons. The negative results are mostly obtained by studying $\widetilde{K}O(X_{n,k})$ and seeing that the tangent bundle does not equal zero here.

I point out here two things. (a) Sam is a problem solver. So am I. I think these are the kinds of things homotopy theorists should be doing. (b) Sam likes to work in teams. Many of his papers involve four authors.

In 1999-2000, he put together a team of four Mexicans to attack parallelizability of complex projective Stiefel manifolds $PW_{n,k}$. Here you mod out by the S^1-action. With Astey, Micha, and Pastor ([7]), he showed that the only parallelizable ones are $PW_{n,n}$ and $PW_{n,n-1}$. The nonexistence was obtained using characteristic classses,

while existence was obtained by identifying the tangent bundle as 0 in $\widetilde{K}O(PW_{n,k})$ and then destabilizing.

6. Interesting algebraic topology text

Finally I mention the text, *Algebraic Topology from a Homotopical Viewpoint*, by Sam together with Marcelo Aguilar and Carlos Prieto, which was published in English by Springer-Verlag in 2002 ([4]). This is a very ambitious and unusual text. The main novelty is *defining* the homology groups of a CW complex X as the homotopy groups of its infinite symmetric product $SP(X)$. Cohomology groups are defined as homotopy classes of maps into Eilenberg-MacLane spaces, which are defined as infinite symmetric products of Moore spaces. Of course, it is a well-known theorem of Dold and Thom ([31]) that $H_*(X)$ is isomorphic to $\pi_*(SP(X))$, but to use this as the definition of homology groups and show that the major properties of homology theory can be developed this way is a bold step requiring much ingenuity.

Roughly, the book divides into thirds:

(1) a fairly standard treatment of fundamental group, homotopy groups, fibrations and cofibrations, and covering spaces;

(2) the novel treatment of homology and cohomology described above;

(3) a thorough and sophisticated treatment of K-theory, characteristic classes, and generalized cohomology.

As I wrote in my review in *Math Reviews*([25]), some topics in homological algebra, such as the definition of Tor and Ext, were not included in the book, instead requiring reference to another text. I feel that a first text should include this material. But, all things considered, I am very impressed that an introductory algebraic topology text can be organized this way. I learned a lot by reading it.

In conclusion: I have outlined many of Sam's important contributions to algebraic topology. Almost as important as the mathematics is the way that Sam has involved others in his work, and has brought younger Mexican mathematicians into algebraic topology in such a way that they have made major contributions. Thank you, Sam, for all this.

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