

# Commutative $C^*$ -algebras of Toeplitz operators and quantization on the unit disk <sup>☆</sup>

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## Abstract

A family of recently discovered commutative  $C^*$ -algebras of Toeplitz operators on the unit disk can be classified as follows. Each pencil of hyperbolic straight lines determines a set of symbols consisting of functions which are constant on the corresponding cycles, the orthogonal trajectories to lines forming a pencil. The  $C^*$ -algebra generated by Toeplitz operators with such symbols turns out to be commutative. We show that these cases are the only possible ones which generate the commutative  $C^*$ -algebras of Toeplitz operators on each weighted Bergman space.

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## 1. Introduction

Let  $\mathbb{D}$  be the unit disk in  $\mathbb{C}$ . Consider  $L_2(\mathbb{D})$  with respect to standard normalized measure and its subspace, the Bergman space  $\mathcal{A}^2(\mathbb{D})$ , which consists of functions analytic in  $\mathbb{D}$ . Let  $B_{\mathbb{D}}$  stand for the orthogonal Bergman projection of  $L_2(\mathbb{D})$  onto  $\mathcal{A}^2(\mathbb{D})$ . Given a function  $a(z) \in L_{\infty}(\mathbb{D})$ , the Toeplitz operator  $T_a$  with *symbol*  $a = a(z)$  is defined on  $\mathcal{A}^2(\mathbb{D})$  as follows:

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$$T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \mapsto B_{\mathbb{D}}(a\varphi) \in \mathcal{A}^2(\mathbb{D}).$$

It is well known (see, for example, [15]) that Toeplitz operators with radial symbols commute and can be diagonalized in the standard monomial basis in  $\mathcal{A}^2(\mathbb{D})$ . Surprisingly (see, for details, [18,19]) aside from radial symbols there exists a rich family of symbols which generate *commutative*  $C^*$ -algebras of Toeplitz operators. Moreover, it turns out that these commutative properties do not depend at all on smoothness properties of symbols: the corresponding symbols can be merely measurable. The prime cause here appears to be the geometric configuration of level lines of symbols. All commutative  $C^*$ -algebras of Toeplitz operators discovered can be classified by pencils of geodesics on the unit disk, considered as the hyperbolic plane. More precise, given a pencil of geodesics, consider the set of symbols constant on the corresponding cycles, the orthogonal trajectories to geodesics forming the pencil. The  $C^*$ -algebra generated by Toeplitz operators with such symbols turns to be commutative. It was shown later (see [8–10]) that the same classes of symbols generate commutative  $C^*$ -algebras of Toeplitz operators on *each weighted Bergman space*.

At the same time the principal question,

*whether the above classes are the only possible sets of symbols which might generate the commutative  $C^*$ -algebras of Toeplitz operators on each weighted Bergman space,*

has remained open.

There is a trivial case having in fact no connection with specific properties of Toeplitz operators. Each  $C^*$ -algebra with identity (Toeplitz operators with the symbol  $e(z) \equiv 1$ ) generated by a self-adjoint element (Toeplitz operator with a real-valued symbol  $a = a(z)$ ) is obviously commutative. We exclude this obvious case from the further considerations.

The aim of the paper is to give the affirmative answer to the above question.

The commutativity of the above algebras on *each weighted Bergman space* is of great importance and permits us to make use of the Berezin quantization procedure (see, for example, [2,3]). At the same time to obtain the necessary information about potential symbols we need to calculate the *second and third terms* in the asymptotic expansion of a commutator. It turns out that the first three terms of this expansion together provide us with exact geometric information: *in order to generate a commutative  $C^*$ -algebra of Toeplitz operators on each weighted Bergman space the symbols must be constant on the cycles of a pencil of geodesics.*

At the same time we show that there exist, in a sense non-typical,  $C^*$ -algebras of Toeplitz operators commutative *only on a single Bergman space*.

The paper is organized as follows.

In Section 2 we introduce Toeplitz operators on the weighted Bergman spaces as well as the pencils of geodesics, and sketch the proof, for a parabolic case, that the  $C^*$ -algebra generated by Toeplitz operators, whose symbols are constant on cycles, is commutative in each weighted Bergman space.

In Section 3 we discuss the symbol classes, clarifying that they have to be linear sets of smooth functions closed under complex conjugation and containing  $e(z) \equiv 1$ .

In Section 4 we show that there exist  $C^*$ -algebras of Toeplitz operators commutative on a *single weighted Bergman space*. However, in the examples given the set of generating symbols is quite restricted.

In Section 5 we recall necessary facts on Berezin quantization on the hyperbolic plane and give the three-term asymptotic expansion of a commutator (of Wick symbols) of Toeplitz operators.

The vanishing of each of these three terms is necessary for a commutativity of the corresponding Toeplitz operator algebra on each weighted Bergman space.

In Section 6 we show that the vanishing of the first term in a commutator implies that all real-valued functions from the symbol class considered must have the same gradient lines and the same level lines.

In Section 7 we show that the vanishing of the second term in a commutator implies that the (common) gradient lines have to be geodesics.

In auxiliary Section 8 we list the description of the curves in  $\mathbb{D}$  whose hyperbolic geodesic curvature is constant and their characterization by the curvature value.

In Section 9 we show that the vanishing of the third term in a commutator implies that the (common) level lines have to be cycles.

Gathering together the results of Sections 6, 7, and 9, we prove in Section 10 our main result:

*a Toeplitz operator algebra is commutative on each weighted Bergman space if and only if the corresponding symbol set consists of (smooth) functions which are constant on the cycles of a pencil of hyperbolic geodesics.*

In final Section 11 we prove the formula for the three term asymptotic expansion of a commutator, announced in Section 5.

## 2. Commutative algebras and hyperbolic geometry

We introduce the following Möbius invariant normalized measure on the unit disk  $\mathbb{D}$ :

$$d\mu(z) = \frac{1}{\pi} \frac{dx \wedge dy}{(1 - (x^2 + y^2))^2} = \frac{1}{2\pi i} \frac{d\bar{z} \wedge dz}{(1 - |z|^2)^2}. \tag{2.1}$$

For  $-1 < \lambda < +\infty$ , the weighted Bergman space  $\mathcal{A}_\lambda^2(\mathbb{D})$  on the unit disk (see, for example, [12]) is the space of analytic functions in  $L_2(\mathbb{D}, d\mu_\lambda)$ , where

$$d\mu_\lambda(z) = (\lambda + 1)(1 - |z|^2)^{\lambda+2} d\mu(z),$$

and

$$\|f\|_\lambda = \left( \int_{\mathbb{D}} |f(z)|^2 d\mu_\lambda(z) \right)^{1/2}.$$

An alternative definition (see, for example, [3]) can be given in terms of another parameter  $h \in (0, 1)$ , for which the weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$  on the unit disk is the space of analytic functions in  $L_2(\mathbb{D}, d\mu_h)$ , where

$$d\mu_h(z) = \left( \frac{1}{h} - 1 \right) (1 - |z|^2)^{1/h} d\mu(z) = \left( \frac{1}{h} - 1 \right) (1 - |z|^2)^{1/h-2} \frac{1}{2\pi i} d\bar{z} \wedge dz.$$

For  $\lambda + 2 = \frac{1}{h}$  we have the same space, and for  $\lambda = 0$  or  $h = \frac{1}{2}$  we have the classical weightless Bergman space  $\mathcal{A}^2(\mathbb{D})$  (with normalized measure).

The orthogonal Bergman projection from  $L_2(\mathbb{D}, d\mu_\lambda)$  or  $L_2(\mathbb{D}, d\mu_h)$  onto the weighted Bergman space has the form (see, for example, [3,12]):

$$(B_{\mathbb{D}}^{(\lambda)} f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{\lambda+2}} d\mu_\lambda(\zeta) = (\lambda + 1) \int_{\mathbb{D}} f(\zeta) \left( \frac{1 - \zeta\bar{z}}{1 - z\bar{\zeta}} \right)^{\lambda+2} d\mu(\zeta),$$

or

$$(B_{\mathbb{D}}^{(h)} f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{1/h}} d\mu_h(\zeta) = \left( \frac{1}{h} - 1 \right) \int_{\mathbb{D}} f(\zeta) \left( \frac{1 - \zeta\bar{z}}{1 - z\bar{\zeta}} \right)^{1/h} d\mu(\zeta). \quad (2.2)$$

Given a function  $a(z) \in L_\infty(\mathbb{D})$ , the Toeplitz operator  $T_a^{(h)}$  with symbol  $a$  is defined on  $\mathcal{A}_h^2(\mathbb{D})$  as follows:

$$T_a^{(h)} : \varphi \in \mathcal{A}_h^2(\mathbb{D}) \mapsto B_{\mathbb{D}}^{(h)}(a\varphi) \in \mathcal{A}_h^2(\mathbb{D}).$$

It was recently shown [18,19] that apart from the known case of radial symbols in the classical (weightless) Bergman space  $\mathcal{A}^2(\mathbb{D})$  there exists a rich family of commutative  $C^*$ -algebras of Toeplitz operators. Moreover, surprisingly it turns out that these commutative properties of Toeplitz operators do not depend at all on smoothness properties of the symbols: the corresponding symbols can be merely measurable. Furthermore, it turns out [8–10] that the above classes of symbols generate commutative  $C^*$ -algebras of Toeplitz operators on each weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$ . The prime cause here appears to be the geometric configuration of level lines of symbols.

In this context it is useful to consider the unit disk  $\mathbb{D}$  as the hyperbolic plane equipped with the standard hyperbolic metric

$$ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}. \quad (2.3)$$

Recall that a geodesic, or a hyperbolic straight line, on  $\mathbb{D}$  is a part of an Euclidean circle or of a straight line orthogonal to the boundary of  $\mathbb{D}$ .

Each pair of geodesics, say  $L_1$  and  $L_2$ , determines (see, for example, [1]) a geometrically defined object, a one-parameter family  $\mathcal{P}$  of geodesics, which is called the pencil defined by  $L_1$  and  $L_2$ . Each pencil has an associated family  $\mathcal{C}$  of lines, called cycles, which are the orthogonal trajectories to geodesics forming the pencil.

The pencil  $\mathcal{P}$  defined by  $L_1$  and  $L_2$  is called:

- (1) *parabolic* if  $L_1$  and  $L_2$  are parallel (and tend to the same point  $z_0 \in \partial\mathbb{D}$ ), in this case  $\mathcal{P}$  is the set of all geodesics parallel to  $L_1$  and  $L_2$ , and the cycles are called *horocycles*;
- (2) *elliptic* if  $L_1$  and  $L_2$  are intersecting (at a point  $z_0 \in \mathbb{D}$ ), in this case  $\mathcal{P}$  is the set of all geodesics passing through the common point of  $L_1$  and  $L_2$ ;
- (3) *hyperbolic* if  $L_1$  and  $L_2$  are disjoint, in this case  $\mathcal{P}$  is the set of all geodesics orthogonal to the unique common orthogonal geodesic (with endpoints  $z_1, z_2 \in \partial\mathbb{D}$ ) of  $L_1$  and  $L_2$ , and the cycles are called *hypercycles*.

In Fig. 1, illustrating possible pencils, the cycles are drawn in bold lines.

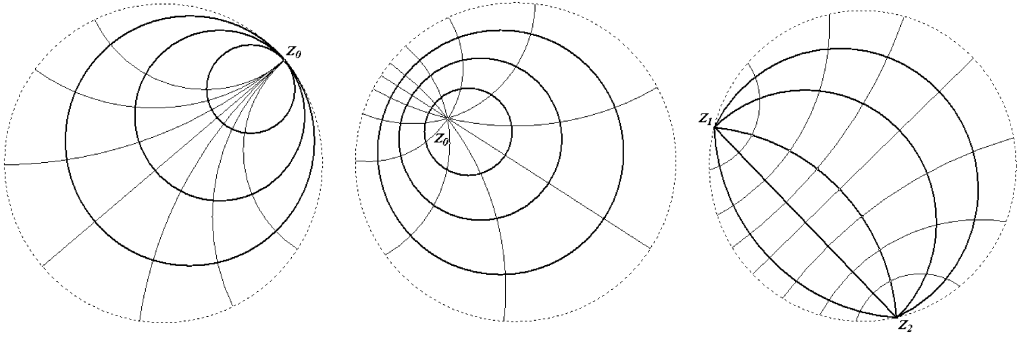


Fig. 1. Parabolic, elliptic and hyperbolic pencils.

The following main result has been proved in [18,19] for the classical (weightless) Bergman space, and in [8–10] for all weighted Bergman spaces.

**Theorem 2.1.** *Given a pencil  $\mathcal{P}$  of geodesics, consider the set of  $L_\infty$ -symbols which are constant on corresponding cycles. The  $C^*$ -algebra generated by Toeplitz operators with such symbols is commutative on each weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$  (or  $\mathcal{A}_\lambda^2(\mathbb{D})$ ).*

For completeness we sketch the proof for the case of a parabolic pencil and a weighted Bergman space  $\mathcal{A}_\lambda^2(\mathbb{D})$ .

Given a parabolic pencil  $\mathcal{P} = \mathcal{P}(z_0)$ , which consists of all parallel lines tending to the point  $z_0 \in \partial\mathbb{D}$ , introduce the Möbius transformation

$$w = \alpha_{z_0}(z) = i \frac{z_0 + z}{z_0 - z} \tag{2.4}$$

of the unit disk  $\mathbb{D}$  onto the upper half-plane  $\Pi$ , which maps the point  $z_0 \in \partial\mathbb{D}$  to the point  $\infty \in \Pi$ . Then the pencil  $\mathcal{P}(\infty) = \alpha_{z_0}(\mathcal{P}(z_0))$  on the upper half-plane  $\Pi$  consists of all semi-lines which are parallel in the Euclidean sense to the positive semi-axis  $\{w = 0 + iv : v \in \mathbb{R}_+\}$ , and the set of all horocycles coincides with the set of all Euclidean straight lines parallel to the real axis being the boundary  $\mathbb{R} = \partial\Pi$  of the upper half-plane. Then the set of all  $L_\infty$ -functions constant on horocycles consists of functions depending only on  $v$ , the imaginary part of  $w = u + iv \in \Pi$ .

Introduce the unitary operators

$$U_1 = \frac{1}{\sqrt{\pi}}(F \otimes I) : L_2(\Pi, d\mu_\lambda) \rightarrow L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}_+, (\lambda + 1)(2y)^\lambda dy),$$

where  $F : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$  is the Fourier transform

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iu\xi} f(\xi) d\xi,$$

and

$$U_2 : L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}_+, (\lambda + 1)(2y)^\lambda dy) \rightarrow L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}_+, dy)$$

which is defined by the rule

$$(U_2\varphi)(x, y) = \frac{1}{\theta_\lambda(|x|)} e^{-y/2+|x|\beta(|x|,y)} \varphi(x, \beta(|x|, y)),$$

where

$$\theta_\lambda(x) = \left( \frac{2x^{\lambda+1}}{(\lambda+1)\Gamma(\lambda+1)} \right)^{1/2}, \quad x \geq 0,$$

and for each fixed  $x > 0$  the function  $\beta(x, y)$  is the inverse function to

$$\gamma(x, t) = -\ln \left\{ \theta_\lambda^2(x)(\lambda+1) \int_t^\infty (2\eta)^\lambda e^{-2x\eta} d\eta \right\}, \tag{2.5}$$

i.e.,  $\beta(x, \gamma(x, t)) = t, x > 0$ . Letting  $\ell_0(y) = e^{-y/2}$ , we have  $\ell_0(y) \in L_2(\mathbb{R}_+)$  and  $\|\ell_0(y)\| = 1$ . Denote by  $L_0$  the one-dimensional subspace of  $L_2(\mathbb{R}_+)$  generated by  $\ell_0(y)$ .

**Theorem 2.2.** *The unitary operator  $U = U_2U_1$  gives an isometric isomorphism of  $L_2(\Pi, d\mu_\lambda)$ , where  $\lambda \in (-1, +\infty)$ , onto  $L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}_+, dy)$  under which the Bergman space  $\mathcal{A}_\lambda^2(\Pi)$  is mapped onto  $L_2(\mathbb{R}_+) \otimes L_0$ .*

$$U : \mathcal{A}_\lambda^2(\Pi) \rightarrow L_2(\mathbb{R}_+) \otimes L_0.$$

Introduce the isometric imbedding

$$R_0 : L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0f)(x, y) = \chi_+(x) f(x) \ell_0(y),$$

where  $\chi_+(x)$  is the characteristic function of the positive half-line. Here the function  $f$  is extended to an element of  $L_2(\mathbb{R})$  by setting  $f(x) \equiv 0$ , for  $x < 0$ . The adjoint operator  $R_0^* : L_2(\Pi) \rightarrow L_2(\mathbb{R}_+)$  is given by

$$(R_0^*\varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x, \eta) \ell_0(\eta) d\eta.$$

The operator  $R_\lambda = R_0^*U$  maps the space  $L_2(\Pi, d\mu_\lambda)$  onto  $L_2(\mathbb{R}_+)$ , and the restriction

$$R_\lambda|_{\mathcal{A}_\lambda^2(\Pi)} : \mathcal{A}_\lambda^2(\Pi) \rightarrow L_2(\mathbb{R}_+)$$

is an isometric isomorphism. The adjoint operator

$$R_\lambda^* = U^*R_0 : L_2(\mathbb{R}_+) \rightarrow \mathcal{A}_\lambda^2(\Pi) \subset L_2(\Pi, d\mu_\lambda)$$

is an isometric isomorphism of  $L_2(\mathbb{R}_+)$  onto the subspace  $\mathcal{A}_\lambda^2(\Pi)$  of the space  $L_2(\Pi, d\mu_\lambda)$ . Moreover, we have

$$R_\lambda R_\lambda^* = I : L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+),$$

$$R_\lambda^* R_\lambda = B_\Pi^\lambda : L_2(\Pi, d\mu_\lambda) \rightarrow \mathcal{A}_\lambda^2(\Pi).$$

Introduce the  $C^*$ -algebra of bounded measurable symbols which depend only on  $v$  (the imaginary part of a variable  $w = u + iv$ )  $\mathcal{A}(\infty) = \mathbb{C} \otimes L_\infty(\mathbb{R}_+) (\subset L_\infty(\mathbb{R} \times \mathbb{R}_+))$ , and consider the Toeplitz operator algebra  $\mathcal{T}_\lambda(\mathcal{A}(\infty))$  generated by Toeplitz operators  $T_a^{(\lambda)}$  with symbols  $a \in \mathcal{A}(\infty)$ .

**Theorem 2.3.** *Let  $a = a(v) \in \mathcal{A}(\infty)$ . Then the Toeplitz operator  $T_a^{(\lambda)}$  acting on  $\mathcal{A}_\lambda^2(\Pi)$  is unitary equivalent to the multiplication operator  $\gamma_{a,\lambda} I = R_\lambda T_a^{(\lambda)} R_\lambda^*$ , acting on  $L_2(\mathbb{R}_+)$ . The function  $\gamma_{a,\lambda}(x)$  is given by*

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(t/2) t^\lambda e^{-xt} dt, \quad x \in \mathbb{R}_+. \tag{2.6}$$

Denote by  $\mathcal{A}(z_0)$  the  $C^*$ -subalgebra of  $L_\infty(\mathbb{D})$  consisting of all functions which are constant on the cycles of the initial pencil  $\mathcal{P}(z_0)$ . The  $C^*$ -algebra  $\mathcal{T}_\lambda(\mathcal{A}(z_0))$  generated by Toeplitz operators with  $\mathcal{A}(z_0)$ -symbols is obviously unitary equivalent to  $\mathcal{T}_\lambda(\mathcal{A}(\infty))$ , and by Theorem 2.3 both of them are commutative. This finishes the proof of Theorem 2.1 for a parabolic pencil.

The proofs for the other two cases, elliptic and hyperbolic, are rather similar. But, instead of the Fourier transform used in the proof for parabolic case, the discrete Fourier transform and the Mellin transform are used for elliptic and hyperbolic cases, respectively.

Recall that any Möbius transformation preserving the unit disk is a movement (conformal isometry) of the hyperbolic plane (the unit disk endowed with the hyperbolic metric), and thus it cannot change the type of a pencil of geodesics. That is neither of these three cases, parabolic, elliptic, and hyperbolic, can be reduced to another one by means of Möbius transformations.

### 3. On symbol classes

Although in many papers dealing with bounded Toeplitz operators the symbol class is assumed to be a Banach or  $C^*$ -algebra, the only a priori natural structure on a set of symbols is a linear space (without any topology). This is quite justly. Indeed, considering bounded Toeplitz operators with unbounded symbols, we gave [11, Example 7] an example of two bounded Toeplitz operators with symbols  $a_1$  and  $a_2$  such that the Toeplitz operator with symbol  $a_1 \cdot a_2$  is unbounded. At the same time, Toeplitz operators with unbounded symbols can easily appear both under the uniform limit of Toeplitz operators with bounded symbols (see Theorem 3.1) and as a result of algebraic operations with Toeplitz operators having bounded symbols (see Lemma 4.1).

Consequently, discussing commutative  $C^*$ -algebras of Toeplitz operators we will always assume that the corresponding generating class of symbols is a linear space. As was already mentioned in the Introduction, there is a trivial case having in fact no connection with specific properties of Toeplitz operators. Each  $C^*$ -algebra with identity (Toeplitz operators with the symbol  $e(z) \equiv 1$ ) generated by a self-adjoint element (Toeplitz operator with a real-valued symbol

$a = a(z)$ ) is obviously commutative. The set of generating symbols here is quite restricted, and coincides with the two-dimensional linear space generated by  $e(z)$  and  $a(z)$ . We exclude this obvious case from further consideration.

To underline the geometric nature of symbol classes which generate the commutative  $C^*$ -algebras of Toeplitz operators we have considered bounded measurable symbols in the previous section. This also agrees with the desire for such (commutative) algebras to be, in a sense, maximal. Note that the arguments used in the proof do not require any assumption on smoothness properties of symbols. The same result (commutativity of Toeplitz operator  $C^*$ -algebra) remains valid for each linear subspace of  $L_\infty$ -symbols (constant on cycles). Moreover, we can start as well with a much more restricted set of symbols (say, smooth symbols only) and extend them furthermore to all  $L_\infty$ -symbols by means of uniform and strong operator limits of sequences of Toeplitz operators. Thus it is irrelevant which class of symbols, smooth or  $L_\infty$ , one starts with.

In Section 6 we assume that the commutative  $C^*$ -algebra of Toeplitz operators is generated by symbols from a certain linear space of smooth functions, and prove later on that the symbols are as described in Section 2, i.e., constant on the cycles of a pencil. Thus, the initial class of symbols can then be extended, knowing the result of Theorem 2.1, to all  $L_\infty$ -functions constant on the cycles of the corresponding pencil.

As a result, in what follows it is sufficient to deal with the smooth symbols only. As we consider the  $C^*$ -algebra generated by Toeplitz operators, we can always assume, without loss of generality, that our set of symbols is closed under complex conjugation and contains the function  $e(z) \equiv 1$ .

To complete the discussion we show how the Toeplitz operators with unbounded symbols can appear under the uniform limit of Toeplitz operators with bounded symbols. In the next theorem we use weighted Bergman spaces  $\mathcal{A}_\lambda^2(\Pi)$  labeled by  $\lambda = \frac{1}{h} - 2$ .

**Theorem 3.1.** *The Toeplitz operator  $T_a^{(\lambda)}$  with unbounded symbol*

$$a(v) = v^{-\beta} \sin v^{-\alpha}, \quad v = \text{Im } w \in \mathbb{R}_+, \tag{3.1}$$

where  $0 < \beta < 1$  and  $\alpha > \beta$ , is bounded and belongs to the  $C^*$ -algebra generated by Toeplitz operators with smooth bounded symbols on each weighted Bergman space  $\mathcal{A}_\lambda^2(\Pi)$ .

**Proof.** First of all by [9, Example 4.4], the Toeplitz operator  $T_a^{(\lambda)}$  with symbol (3.1) is bounded on each  $\mathcal{A}_\lambda^2(\Pi)$ , where  $\lambda \geq 0$ .

Consider now the sequence of symbols  $\{a_n\}$ , where

$$a_n(v) = \begin{cases} a(v), & v \in [v_n, \infty), \\ 0, & v \in [0, v_n), \end{cases}$$

and  $v_n = (\pi n)^{-1/\alpha}$  are zeros of function (3.1). Note that each symbol  $a_n(v)$  is bounded and continuous. Further each one can be uniformly approximated by smooth symbols and thus belongs to the  $C^*$ -algebra generated by Toeplitz operators with smooth bounded symbols. By Theorem 2.3 the Toeplitz operator  $T_a^{(\lambda)}$  acting on  $\mathcal{A}_\lambda^2(\Pi)$  is unitary equivalent to the multiplication operator  $\gamma_{a,\lambda} I$  acting on  $L_2(\mathbb{R}_+)$ , where the function  $\gamma_{a,\lambda}(x)$  is given by (2.3). Thus



$$\begin{aligned} \|T_a^{(\lambda)} - T_{a_n}^{(\lambda)}\| &= \|T_{(a-a_n)}^{(\lambda)}\| \\ &= \sup_{x \in \mathbb{R}_+} |\gamma_{(a-a_n), \lambda}(x)| = \sup_{x \in \mathbb{R}_+} \left| \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^{v_n} a(t/2) t^\lambda e^{-xt} dt \right| \\ &= \sup_{x \in \mathbb{R}_+} \left| \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \left( B_{a,0}^{(1)}(v_n) v_n^\lambda e^{-xv_n} - \int_0^{v_n} B_{a,0}^{(1)}(t) (\lambda - xt) t^{\lambda-1} e^{-xt} dt \right) \right|, \end{aligned}$$

where

$$B_{a,0}^{(1)}(t) = \int_0^t a(v/2) dv.$$

By [9, Eq. (4.11)], we have

$$|B_{a,0}^{(1)}(t)| \leq \text{const } t^{\alpha-\beta+1},$$

where the constant does not depend on  $t \in (0, 1)$ . Thus

$$\begin{aligned} \|T_{(a-a_n)}^{(\lambda)}\| &\leq \text{const} \sup_{x \in \mathbb{R}_+} \left| \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \left( v_n^{\alpha-\beta+1+\lambda} e^{-xv_n} + \lambda \int_0^{v_n} t^{\alpha-\beta+\lambda} e^{-xt} dt \right. \right. \\ &\quad \left. \left. + x \int_0^{v_n} t^{\alpha-\beta+\lambda+1} e^{-xt} dt \right) \right| \\ &\leq \text{const} \frac{1}{\Gamma(\lambda+1)} \sup_{x \in \mathbb{R}_+} (v_n^{\alpha-\beta} (v_n x)^{\lambda+1} e^{-v_n x}) \\ &\quad + \text{const} \frac{1}{\Gamma(\lambda+1)} \sup_{x \in \mathbb{R}_+} \left( \frac{\lambda}{x^{\alpha-\beta}} \int_0^{v_n x} v^{\alpha-\beta+\lambda} e^{-v} dv \right) \\ &\quad + \text{const} \frac{1}{\Gamma(\lambda+1)} \sup_{x \in \mathbb{R}_+} \left( \frac{1}{x^{\alpha-\beta}} \int_0^{v_n x} v^{\alpha-\beta+\lambda+1} e^{-v} dv \right) \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

To evaluate  $I_1$  note that

$$\sup_{x \in \mathbb{R}_+} ((v_n x)^{\lambda+1} e^{-v_n x}) < \infty,$$

thus

$$I_1 \leq c_1(\lambda) v_n^{\alpha-\beta}.$$

Evaluating  $I_2$  assume first that  $v_n x \leq 1$ , then

$$\begin{aligned}
 I_2 &\leq \text{const} \frac{1}{\Gamma(\lambda + 1)} \sup_{x \in \mathbb{R}_+} \frac{\lambda}{x^{\alpha-\beta}} \cdot (v_n x)^{\alpha-\beta+\lambda+1} \\
 &\leq c_2(\lambda) \sup_{x \in \mathbb{R}_+} v_n^{\alpha-\beta} \cdot (v_n x)^{\lambda+1} \leq c_2(\lambda) v_n^{\alpha-\beta}.
 \end{aligned}$$

If  $v_n x > 1$ , that is  $x > v_n^{-1}$ , we have

$$I_2 \leq \text{const} \frac{\lambda}{\Gamma(\lambda + 1)} v_n^{\alpha-\beta} \int_0^\infty v^{\alpha-\beta+\lambda} e^{-v} dv \leq c_3(\lambda) v_n^{\alpha-\beta}.$$

The evaluation of  $I_3$  is quite analogous. Thus we have

$$\|T_a^{(\lambda)} - T_{a_n}^{(\lambda)}\| \leq c(\lambda) \cdot v_n^{\alpha-\beta},$$

where the constant  $c(\lambda)$  depends on  $\lambda$  but does not depend on  $n$ , and where  $v_n$  tends to 0 as  $n$  tends to infinity.  $\square$

#### 4. Commutativity on a single Bergman space

Before passing to Toeplitz operator algebras commutative on every weighted Bergman space, we present results on the problem:

*whether there exist  $C^*$ -algebras of Toeplitz operators commutative on a single Bergman space.*

We start with the Toeplitz operator  $C^*$ -algebra with identity generated by the (single) Toeplitz operator with symbol  $a(z) = \text{Re } z = x$ . This algebra is commutative on each weighted Bergman space  $\mathcal{A}_\lambda^2(\mathbb{D})$ . Observe now that it contains Toeplitz operators different from linear combinations of the initial generators.

**Lemma 4.1.** *Given a weighted Bergman space  $\mathcal{A}_\lambda^2(\mathbb{D})$ , the following equality holds:*

$$(T_x^{(\lambda)})^2 = T_{x^2}^{(\lambda)} + K^{(\lambda)},$$

where the compact operator  $K^{(\lambda)} = T_{k_\lambda(r)}^{(\lambda)}$  is the Toeplitz operator with a certain radial symbol  $k_\lambda(r)$ ,  $r = |z|$ . In particular,

$$\begin{aligned}
 k_0(r) &= \frac{1}{4}(1 - r^2 + \ln r^2), \\
 k_1(r) &= -\frac{1}{4}\left(1 + r^2 + \frac{2r^2}{1 - r^2} \ln r^2\right).
 \end{aligned}$$

**Proof.** Consider the following orthogonal (not orthonormal) basis, common for all weighted Bergman spaces  $\mathcal{A}_\lambda^2(\mathbb{D})$ :

$$e_n(z) = z^n, \quad n = 0, 1, 2, \dots$$

We prove first that the operator  $T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)}$  is diagonal with respect to this basis, and, moreover,

$$(T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)} e_n)(z) = \frac{n}{n + \lambda + 1} e_n(z). \tag{4.1}$$

We have obviously

$$(T_z^{(\lambda)} e_n)(z) = e_{n+1}(z). \tag{4.2}$$

Calculate now

$$(T_{\bar{z}}^{(\lambda)} e_n)(z) = \frac{\lambda + 1}{\pi} \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^\lambda \bar{\zeta} \zeta^n}{(1 - z\bar{\zeta})^{\lambda+2}} d\nu(\zeta),$$

where  $d\nu$  is the usual Lebesgue plane measure. Changing the variable  $\zeta = rt$ , where  $r \in [0, 1]$  and  $t$  belongs to the unit circle  $\mathbb{T}$ , we have

$$(T_{\bar{z}}^{(\lambda)} e_n)(z) = \frac{\lambda + 1}{\pi i} \int_0^1 \int_{\mathbb{T}} \frac{(1 - r^2)^\lambda r^{n+2} t^{n-2}}{(1 - zrt^{-1})^{\lambda+2}} dt dr.$$

Substitute

$$(1 - zrt^{-1})^{-(\lambda+2)} = \sum_{k=0}^{\infty} (-1)^k C_k^{-(\lambda+2)} (zrt^{-1})^k,$$

where

$$(-1)^k C_k^{-(\lambda+2)} = \frac{(\lambda + 2)(\lambda + 3) \dots (\lambda + k + 1)}{k!}.$$

Then according to residue theory we have

$$\begin{aligned} (T_{\bar{z}}^{(\lambda)} e_n)(z) &= 2(\lambda + 1)(-1)^{n-1} C_{n-1}^{-(\lambda+2)} z^{n-1} \int_0^1 (1 - r^2)^\lambda r^{2n+1} dr \\ &= (\lambda + 1)(-1)^{n-1} C_{n-1}^{-(\lambda+2)} z^{n-1} \int_0^1 (1 - r)^\lambda r^n dr \\ &= (\lambda + 1)(-1)^{n-1} C_{n-1}^{-(\lambda+2)} \mathbf{B}(\lambda + 1, n + 1) z^{n-1} \end{aligned}$$

$$\begin{aligned}
 &= (\lambda + 1) \frac{(\lambda + 2)(\lambda + 3) \dots (\lambda + n)}{(n - 1)!} \frac{\Gamma(\lambda + 1)\Gamma(n + 1)}{\Gamma(\lambda + n + 2)} z^{n-1} \\
 &= \frac{n}{n + \lambda + 1} e_{n-1}(z),
 \end{aligned} \tag{4.3}$$

where  $B$  and  $\Gamma$  are classical Gauss functions. Now from (4.2) and (4.3) we have (4.1).

Recall that each Toeplitz operator with radial symbol, as well as the operator  $T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)}$ , is diagonal in the basis  $\{e_n(z)\}$ . Moreover (see, for example, [6]), we have

$$T_z^{(0)} T_{\bar{z}}^{(0)} = T_{\tilde{k}_0(r)}^{(0)}, \quad \text{where } \tilde{k}_0(r) = 1 + \ln r^2. \tag{4.4}$$

In fact, for each  $\lambda$ , the operator  $T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)}$  is the Toeplitz operator with a certain radial symbol  $\tilde{k}_\lambda(r)$ . We do not present the exact formula for  $\tilde{k}_\lambda(r)$  here, but mention, for example, that

$$\tilde{k}_1(r) = -\left(1 + \frac{2r^2}{1 - r^2} \ln r^2\right). \tag{4.5}$$

Formulas (4.4) and (4.5) can be easily checked by comparing the spectral sequences given by (4.1), on the one hand, and by (4.5) (see [10, formula (2.1)]) from the other.

Calculate now

$$\begin{aligned}
 (T_x^{(\lambda)})^2 &= \frac{1}{4} (T_z^{(\lambda)} + T_{\bar{z}}^{(\lambda)})^2 = \frac{1}{4} (T_z^{(\lambda)} T_z^{(\lambda)} + T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)} + T_{\bar{z}}^{(\lambda)} T_z^{(\lambda)} + T_{\bar{z}}^{(\lambda)} T_{\bar{z}}^{(\lambda)}) \\
 &= \frac{1}{4} (T_{z^2}^{(\lambda)} + T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)} + T_{|z|^2}^{(\lambda)} + T_{\bar{z}^2}^{(\lambda)}) \\
 &= T_{x^2}^{(\lambda)} + \frac{1}{4} (T_z^{(\lambda)} T_{\bar{z}}^{(\lambda)} - T_{r^2}^{(\lambda)}) = T_{x^2}^{(\lambda)} + K^{(\lambda)}.
 \end{aligned} \tag{4.6}$$

The operator  $K^{(\lambda)}$  is obviously compact, and by the above is a Toeplitz operator with a certain radial symbol  $k_\lambda(r)$ . Moreover, by (4.4) and (4.5) we have

$$\begin{aligned}
 k_0(r) &= \frac{1}{4} (1 - r^2 + \ln r^2), \\
 k_1(r) &= -\frac{1}{4} \left(1 + r^2 + \frac{2r^2}{1 - r^2} \ln r^2\right). \quad \square
 \end{aligned}$$

Let  $\mathcal{A}(\mathbb{D})$  be a set of symbols. By  $\mathcal{T}_\lambda(\mathcal{A}(\mathbb{D}))$  we denote the  $C^*$ -algebra generated by Toeplitz operators with symbols from  $\mathcal{A}(\mathbb{D})$ , acting on the weighted Bergman space  $\mathcal{A}_\lambda^2(\mathbb{D})$ .

**Theorem 4.2.** *Given any weighted Bergman space  $\mathcal{A}_{\lambda_0}^2(\mathbb{D})$ , where  $\lambda_0 \in (-1, +\infty)$ , there exists a set of symbols  $\mathcal{A}_{\lambda_0}(\mathbb{D})$  such that the Toeplitz operator  $C^*$ -algebra  $\mathcal{T}_{\lambda_0}(\mathcal{A}_{\lambda_0}(\mathbb{D}))$  is commutative, while all others algebras  $\mathcal{T}_\lambda(\mathcal{A}_{\lambda_0}(\mathbb{D}))$ ,  $\lambda \neq \lambda_0$ , are noncommutative.*

**Proof.** We prove the theorem for  $\lambda_0 = 0$ , all other  $\lambda$  are treated quite analogously. Introduce the set  $\mathcal{A}_0(\mathbb{D})$  as the linear space generated by the following tree functions:

$$e(z) \equiv 1, \quad a(z) = \operatorname{Re} z = x, \quad a_0(z) = x^2 + k_0(r) = x^2 + \frac{1}{4} (1 - r^2 + \ln r^2). \tag{4.7}$$

The algebra  $\mathcal{T}_0(\mathcal{A}_0(\mathbb{D}))$  is obviously commutative, since by Lemma 4.1  $T_{a_0}^{(0)} = (T_x^{(0)})^2$ .

To finish the proof we show that

$$T_x^{(\lambda)} T_{a_0}^{(\lambda)} \neq T_{a_0}^{(\lambda)} T_x^{(\lambda)}, \tag{4.8}$$

for each  $\lambda \neq 0$ . Since the operators  $T_x^{(\lambda)}$  and  $(T_x^{(\lambda)})^2$  obviously commute, we have that (4.8) is equivalent to

$$T_x^{(\lambda)} (T_{a_0}^{(\lambda)} - (T_x^{(\lambda)})^2) \neq (T_{a_0}^{(\lambda)} - (T_x^{(\lambda)})^2) T_x^{(\lambda)},$$

or by (4.6) to

$$T_x^{(\lambda)} (T_{\tilde{k}_0}^{(\lambda)} - T_z^{(\lambda)} T_{\tilde{z}}^{(\lambda)}) \neq (T_{\tilde{k}_0}^{(\lambda)} - T_z^{(\lambda)} T_{\tilde{z}}^{(\lambda)}) T_x^{(\lambda)}. \tag{4.9}$$

By (4.2) and (4.3) we have

$$(T_x^{(\lambda)} e_n)(z) = \frac{1}{2} ((T_z^{(\lambda)} + T_{\tilde{z}}^{(\lambda)}) e_n)(z) = \frac{1}{2} e_{n+1}(z) + \frac{n}{2(n + \lambda + 1)} e_{n-1}(z).$$

The Toeplitz operator  $T_{\tilde{k}_0(r)}^{(\lambda)}$  is diagonal in the basis  $\{e_n(z)\}$ , and, moreover, by [10, formula (2.1)], we have

$$(T_{\tilde{k}_0(r)}^{(\lambda)} e_n)(z) = \gamma_{\tilde{k}_0, \lambda}(n) e_n(z), \tag{4.10}$$

where

$$\begin{aligned} \gamma_{\tilde{k}_0, \lambda}(n) &= \frac{1}{\mathbf{B}(n + 1, \lambda + 1)} \int_0^1 \tilde{k}_0(\sqrt{r}) (1 - r)^\lambda r^n dr \\ &= 1 + \frac{1}{\mathbf{B}(n + 1, \lambda + 1)} \int_0^1 (1 - r)^\lambda r^n \ln r dr \\ &= 1 + [\psi(n + 1) - \psi(n + \lambda + 2)]. \end{aligned}$$

Passing to the last equality we use formula 4.253 from [7], where  $\psi(x) = \Gamma'(x)/\Gamma(x)$  is the so called psi function.

We prove now that the two sides of (4.9) are different, for example, on the first basis element  $e_0(z)$ . Indeed, by the above

$$I_1 := (T_x^{(\lambda)} (T_{\tilde{k}_0}^{(\lambda)} - T_z^{(\lambda)} T_{\tilde{z}}^{(\lambda)}) e_0)(z) = \frac{1}{2} (1 + [\psi(1) - \psi(\lambda + 2)]) e_1(z),$$

while

$$I_2 := ((T_{\tilde{k}_0}^{(\lambda)} - T_z^{(\lambda)} T_{\tilde{z}}^{(\lambda)}) T_x^{(\lambda)} e_0)(z) = \frac{1}{2} \left( 1 + [\psi(2) - \psi(\lambda + 3)] - \frac{1}{\lambda + 2} \right) e_1(z).$$

Note that  $\psi(x + 1) = \psi(x) + \frac{1}{x}$ . Thus

$$\begin{aligned} I_1 - I_2 &= \frac{1}{2} \left( \psi(1) - \psi(\lambda + 2) - \psi(2) + \psi(\lambda + 3) + \frac{1}{\lambda + 2} \right) e_1(z) \\ &= -\frac{\lambda}{2(\lambda + 2)} e_1(z), \end{aligned}$$

which is nonzero if and only if  $\lambda \neq 0$ .  $\square$

### 5. Berezin quantization on the unit disk

We start with recalling the necessary ingredients of the Berezin quantization (see, for example, [2,3]) on the hyperbolic plane. As a classical mechanics we consider the pair  $(\mathbb{D}, \omega)$ , where  $\mathbb{D}$  is the unit disk and the symplectic form  $\omega$  is given by (2.1):

$$\omega = d\mu(z) = \frac{1}{\pi} \frac{dx \wedge dy}{(1 - (x^2 + y^2))^2} = \frac{1}{2\pi i} \frac{d\bar{z} \wedge dz}{(1 - |z|^2)^2}. \tag{5.1}$$

Given two functions  $a, b \in C^\infty(\mathbb{D})$ , their Poisson bracket is as follows:

$$\begin{aligned} \{a, b\} &= \pi (1 - (x^2 + y^2))^2 \left( \frac{\partial a}{\partial y} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} \right) \\ &= 2\pi i (1 - z\bar{z})^2 \left( \frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} - \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} \right). \end{aligned} \tag{5.2}$$

Recall that the Laplace–Beltrami operator has the form

$$\Delta = \pi (1 - (x^2 + y^2))^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = 4\pi (1 - z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}}. \tag{5.3}$$

Let  $E = (0, \frac{1}{2\pi})$ , for each  $\hbar \in E$ , and consequently  $h \in (0, 1)$ , introduce the Hilbert space  $H_h$  as the weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$ . For each function  $a = a(z) \in C^\infty(\mathbb{D})$  consider the family of Toeplitz operators  $T_a^{(h)}$  with (anti-Wick) symbol  $a$  acting on  $\mathcal{A}_h^2(\mathbb{D})$ , for  $h \in (0, 1)$ , and denote by  $\mathcal{T}_h$  the \*-algebra generated by Toeplitz operators  $T_a^{(h)}$  with symbols  $a \in C^\infty(\mathbb{D})$ .

The Wick symbols of the Toeplitz operator  $T_a^{(h)}$  has the form

$$\tilde{a}_h(z, \bar{z}) = (S_h a)(z, \bar{z}) = \left( \frac{1}{h} - 1 \right) \int_{\mathbb{D}} a(\zeta) \left( \frac{(1 - |z|^2)(1 - |\zeta|^2)}{(1 - z\bar{\zeta})(1 - \zeta\bar{z})} \right)^{1/h} d\mu(\zeta). \tag{5.4}$$

For each  $h \in (0, 1)$  we define the function algebra  $\tilde{\mathcal{A}}_h$  generated by Wick symbols  $\tilde{a}_h(z, \bar{z})$  of Toeplitz operators  $T_a^{(h)}$  with smooth (anti-Wick) symbol  $a \in C^\infty(\mathbb{D})$ ; the algebra  $\tilde{\mathcal{A}}_h$  has point-wise linear operations, and the multiplication law (star product) is defined as follows:

$$(\tilde{a}_h \star \tilde{b}_h)(z, \bar{z}) = \left( \frac{1}{h} - 1 \right) \int_{\mathbb{D}} \tilde{a}_h(z, \bar{\zeta}) \tilde{b}_h(\zeta, \bar{z}) \left( \frac{(1 - |z|^2)(1 - |\zeta|^2)}{(1 - z\bar{\zeta})(1 - \zeta\bar{z})} \right)^{1/h} d\mu(\zeta). \tag{5.5}$$

Under such operations the function algebra  $\tilde{\mathcal{A}}_h$  and the Toeplitz operator algebra  $\mathcal{T}_h$  are isomorphic.

The correspondence principle is given by

$$\begin{aligned} \tilde{a}_h(z, \bar{z}) &= a(z, \bar{z}) + O(\hbar), \\ (\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) &= i\hbar\{a, b\} + O(\hbar^2). \end{aligned} \tag{5.6}$$

Formula (5.6) immediately leads to a certain information about the symbols which might generate a commutative Toeplitz operator algebra.

**Theorem 5.1.** *Let  $\mathcal{A}(\mathbb{D})$  be a subalgebra of  $C^\infty(\mathbb{D})$  such that for each  $h \in (0, 1)$  the Toeplitz operator algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ , i.e., the  $C^*$ -algebra generated by the operators  $T_a^{(h)}$ , with  $a \in \mathcal{A}(\mathbb{D})$ , acting on the weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$ , is commutative. Then  $\mathcal{A}(\mathbb{D})$  is a commutative Lie algebra, i.e.,  $\{a, b\} = 0$  for all  $a, b \in \mathcal{A}(\mathbb{D})$ .*

**Proof.** The commutativity of the Toeplitz operator algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  implies that the subalgebra  $\tilde{\mathcal{A}}_h = \{\tilde{a}_h: a \in \mathcal{A}(\mathbb{D})\}$  of the algebra  $\tilde{\mathcal{A}}_h$  is commutative as well. By (5.6) for all  $a, b \in \mathcal{A}(\mathbb{D})$  we have

$$\frac{i}{2\pi}\{a, b\} = \lim_{h \rightarrow 0} \frac{1}{h}(\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h) = 0. \quad \square$$

The geometric information which follows from (5.6) and Theorem 5.1 is insufficient for our purposes. Our main results will follow from the second and third terms of the asymptotic expansion of the commutator of two Wick symbols, which are given by the following theorem.

**Theorem 5.2.** *For any pair  $a = a(z, \bar{z})$  and  $b = b(z, \bar{z})$  of six times continuously differentiable functions the following three-term asymptotic expansion formula holds:*

$$\begin{aligned} \tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h &= \frac{i\hbar}{2\pi}\{a, b\} \\ &+ \frac{\hbar^2}{2} \left[ \frac{i}{8\pi^2}(\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) + \frac{i}{\pi}\{a, b\} \right] \\ &+ \hbar^3 \left[ \frac{i}{192\pi^3}(\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\}) \right. \\ &+ \Delta^2\{a, b\} + \Delta\{a, \Delta b\} + \Delta\{\Delta a, b\} \\ &+ \left. \frac{7i}{48\pi^2}(\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) + \frac{i}{2\pi}\{a, b\} \right] + o(\hbar^3) \\ &= i\hbar\{a, b\} + i\frac{\hbar^2}{4}(\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\} + 8\pi\{a, b\}) \\ &+ i\frac{\hbar^3}{24}[\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2\{a, b\} \\ &+ \Delta\{a, \Delta b\} + \Delta\{\Delta a, b\} + 28\pi(\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) \\ &+ 96\pi^2\{a, b\}] + o(\hbar^3), \end{aligned}$$

where the Poisson bracket  $\{, \}$  and the Laplace–Beltrami operator  $\Delta$  in coordinates  $(z, \bar{z})$  are given by (5.2) and by (5.3), respectively.

The proof of the theorem is rather technical and will be given in final Section 11.

**Corollary 5.3.** *Let  $\mathcal{A}(\mathbb{D})$  be a subalgebra of  $C^\infty(\mathbb{D})$  such that for each  $h \in (0, 1)$  the Toeplitz operator algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  is commutative. Then for all  $a, b \in \mathcal{A}(\mathbb{D})$  we have*

$$\{a, b\} = 0, \tag{5.7}$$

$$\{a, \Delta b\} + \{\Delta a, b\} = 0, \tag{5.8}$$

$$\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0. \tag{5.9}$$

### 6. First term: common gradient and level lines

We start by introducing the symbol classes to be used. Let  $\mathcal{A}(\mathbb{D})$  be a linear space of (smooth) functions. Denote by  $\mathcal{T}(\mathcal{A}(\mathbb{D})) = \{\mathcal{T}_h(\mathcal{A}(\mathbb{D}))\}_h$  the family of  $C^*$ -algebras  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  generated by Toeplitz operators with symbols from  $\mathcal{A}(\mathbb{D})$  and acting on the weighted Bergman spaces  $\mathcal{A}_h^2(\mathbb{D})$ .

We will call  $\mathcal{A}(\mathbb{D})$  a generating space of symbols. In principle the same family can be generated by different generating spaces, i.e., there may exist different  $\mathcal{A}_1(\mathbb{D})$  and  $\mathcal{A}_2(\mathbb{D})$  such that  $\mathcal{T}(\mathcal{A}_1(\mathbb{D})) = \mathcal{T}(\mathcal{A}_2(\mathbb{D}))$ .

Note, that the important thing here is that the generating set must be the same for all values of the parameter  $h$ .

We will call the family  $\mathcal{T} = \{\mathcal{T}_h\}_h$  a single generated family if among its generating spaces there is a two-dimensional space  $\mathcal{A}_a(\mathbb{D})$  which is generated by  $e(z) \equiv 1$  and a real-valued function  $a(z)$ .

Consider in this connection the following two examples.

**Example 6.1.** Let  $a(z) = \operatorname{Re} z = x$ . Consider the corresponding single generated family of Toeplitz operator algebras  $\mathcal{T}(\mathcal{A}_x(\mathbb{D})) = \{\mathcal{T}_h(\mathcal{A}_x(\mathbb{D}))\}_h$ . As it was shown at the proof of Theorem 4.2, the algebra  $\mathcal{T}_{h_0}(\mathcal{A}_x(\mathbb{D}))$  with  $h_0 = 1/2$  is generated as well by  $\mathcal{A}(\mathbb{D})$  which is a linear span of the three functions of (4.7). At the same time none of the other algebras  $\mathcal{T}_h(\mathcal{A}_x(\mathbb{D}))$  is generated by  $\mathcal{A}(\mathbb{D})$ .

**Example 6.2.** Let  $a(z) = |z\bar{z}|^{1/2} = r$ . Consider the corresponding single generated family of Toeplitz operator algebras  $\mathcal{T}(\mathcal{A}_r(\mathbb{D})) = \{\mathcal{T}_h(\mathcal{A}_r(\mathbb{D}))\}_h$ . It can be proved that the family  $\mathcal{T}(\mathcal{A}_r(\mathbb{D}))$  can be generated as well by many other linear spaces of radial functions  $a(r) \in C([0, 1])$  which contain  $e(z) \equiv 1$  and are closed under the complex conjugation, and in particular by the whole  $C([0, 1])$ . At the same time, it can be proved that the family  $\mathcal{T} = \{\mathcal{T}_h\}_h$ , which is generated by the linear space of all smooth radial functions (not necessarily continuous at 1), is wider and cannot be generated by  $\mathcal{A}_r(\mathbb{D})$ .

In what follows we will consider families of commutative Toeplitz operator algebras which contain among their generating spaces the ones given by Definition 6.3. To introduce them we need to consider the notion of the jet of a function, see, for example, [14,17]. Given two complex-valued smooth functions  $f$  and  $g$  defined in a neighborhood of a point  $z \in \mathbb{D}$ , we say that they



have the same jet of order  $k$  at  $z$  if their real partial derivatives at  $z$  up to order  $k$  are equal. It is easy to see that such relation does not depend on the coordinate system and that it defines an equivalence relation. The corresponding equivalence class of a function  $f$  at  $z$  is denoted by  $j_z^k(f)$  and is called the  $k$ th order jet of  $f$  at  $z$ . Furthermore, given a complex vector space  $\mathcal{A}(\mathbb{D})$  of smooth functions, we denote with  $J_z^k(\mathcal{A}(\mathbb{D}))$  the space of  $k$ -jets at  $z$  of the elements in  $\mathcal{A}(\mathbb{D})$ . We observe that  $J_z^k(\mathcal{A}(\mathbb{D}))$  is a finite-dimensional complex vector space.

In what follows, for a differentiable function  $f : \mathbb{D} \rightarrow \mathbb{C}$  we will say that  $z \in \mathbb{D}$  is a nonsingular point of  $f$  if  $df_z \neq 0$ .

The symbol classes that we want to consider are given in the next definition.

**Definition 6.3.** Let  $\mathcal{A}(\mathbb{D})$  be a complex vector space of smooth functions. We will say that  $\mathcal{A}(\mathbb{D})$  is  $k$ -rich if it is closed under complex conjugation and the following conditions are satisfied:

- (i) there is a finite set  $S$  such that for every  $z \in \mathbb{D} \setminus S$  at least one element of  $\mathcal{A}(\mathbb{D})$  is nonsingular at  $z$ ,
- (ii) for every point  $z \in \mathbb{D} \setminus S$  and  $l = 0, \dots, k$ , the space of jets  $J_z^l(\mathcal{A}(\mathbb{D}))$  has complex dimension at least  $l + 1$ .

Observe that  $k$ -richness implies  $l$ -richness for  $l \leq k$ . The following result ensures that  $k$ -richness, for each  $k \geq 2$ , excludes from consideration commutative Toeplitz  $C^*$ -algebras with identity generated by a single self-adjoint Toeplitz operator.

**Lemma 6.4.** Let  $\mathcal{A}(\mathbb{D})$  be a 2-rich space of smooth functions. Then, there is no open set  $V$  in  $\mathbb{D}$  such that the restriction  $\mathcal{A}(\mathbb{D})|_V$  is generated by a single real-valued function  $a \in \mathcal{A}(\mathbb{D})$  and  $e(z) \equiv 1$ .

**Proof.** Suppose such real-valued function  $a$  exists for some open set  $V$ . Then for every  $b \in \mathcal{A}(\mathbb{D})|_V$  there exist  $c_1, c_2 \in \mathbb{C}$  such that  $b = c_1 a + c_2 e$ . Thus  $j_z^2(b) = c_1 j_z^2(a) + c_2 j_z^2(e)$ , for each  $z \in V$ , or the complex dimension of  $J_z^2(\mathcal{A}(\mathbb{D}))$  is at most 2.  $\square$

The previous lemma shows the importance of considering a symbol set  $\mathcal{A}(\mathbb{D})$  which is at least 2-rich. Having such a set  $\mathcal{A}(\mathbb{D})$ , assume now that the Toeplitz operator algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  is commutative for each  $h \in (0, 1)$ . Then by Corollary 5.3 we have that for all  $a, b \in \mathcal{A}(\mathbb{D})$  equalities (5.7), (5.8), and (5.9) must be satisfied.

Note that, as the set  $\mathcal{A}(\mathbb{D})$  is closed under the complex conjugation, it is sufficient to consider conditions (5.7), (5.8), and (5.9) for real-valued functions only.

Each real-valued function  $a \in \mathcal{A}(\mathbb{D})$  with nonvanishing gradient in an open set  $U \in \mathbb{D}$ , has in  $U$  two systems of mutually orthogonal smooth lines, the system of level lines and the system of gradient lines. Given any such a pair, a function  $a$  and an open set  $U$ , it is easy to see that the two above systems of lines can be parameterized to be a new orthogonal coordinate system  $(u, v)$  in  $U$ . The level lines and the gradient lines of the function  $a$  in the coordinates  $(u, v)$  are given respectively as

$$u = u_0 = \text{const} \quad \text{and} \quad v = v_0 = \text{const}.$$

Thus, in particular, we have  $a = a(u) = a(u(x, y))$ .

The coordinate systems  $(u, v)$  and  $(x, y)$  are connected by

$$u = u(x, y), \quad v = v(x, y), \quad \text{or} \quad x = x(u, v), \quad y = y(u, v),$$

with

$$D = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \neq 0,$$

and the orthogonality of the coordinate system  $(u, v)$  is equivalent to

$$\frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \equiv 0.$$

In the coordinates  $(u, v)$  the metric, the symplectic form, and the Poisson brackets have respectively the following form:

$$ds^2 = \tilde{g}_{11}(u, v) du^2 + \tilde{g}_{22}(u, v) dv^2,$$

where

$$\tilde{g}_{11} = g(x, y) \left[ \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial y}{\partial u} \right)^2 \right], \quad \tilde{g}_{22} = g(x, y) \left[ \left( \frac{\partial x}{\partial v} \right)^2 + \left( \frac{\partial y}{\partial v} \right)^2 \right],$$

with  $g = g(x, y) = \pi^{-1}(1 - (x^2 + y^2))^{-2}$ , and

$$\begin{aligned} \omega &= g(x, y) D du \wedge dv, \\ \{f_1, f_2\} &= g^{-1}(x, y) D \left( \frac{\partial f_1}{\partial v} \frac{\partial f_2}{\partial u} - \frac{\partial f_1}{\partial u} \frac{\partial f_2}{\partial v} \right). \end{aligned} \tag{6.1}$$

The geometric information contained in the first term of asymptotic expansion of a commutator, or equivalently in condition (5.7), is given by the next lemma.

**Lemma 6.5.** *Let  $\mathcal{A}(\mathbb{D})$  be a 2-rich space of smooth functions which generates for each  $h \in (0, 1)$  the commutative  $C^*$ -algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  of Toeplitz operators. Then all real-valued functions in  $\mathcal{A}(\mathbb{D})$  have (globally) the same set of level lines and the same set of gradient lines.*

**Proof.** Fix a real-valued function  $a \in \mathcal{A}(\mathbb{D})$  and the local orthogonal coordinate system as above. For any other (real-valued) function  $b \in \mathcal{A}(\mathbb{D})$  condition (5.7) implies

$$\{a, b\} = -g^{-1} Da'(u) \frac{\partial b}{\partial v} \equiv 0,$$

or

$$\frac{\partial b}{\partial v} \equiv 0,$$

that is the function  $b$  has (in  $U_0$ ) the same level lines, and thus has the same gradient lines as the function  $a$ .  $\square$

### 7. Second term: gradient lines are geodesics

We keep considering the above (local) orthogonal coordinate system  $(u, v)$ . All real-valued functions from  $\mathcal{A}(\mathbb{D})$  have the same set of level lines:

$$u = u_0 = \text{const}, \quad \text{or} \quad \begin{cases} x = x(u_0, v), \\ y = y(u_0, v), \end{cases}$$

and the same set of gradient lines:

$$v = v_0 = \text{const}, \quad \text{or} \quad \begin{cases} x = x(u, v_0), \\ y = y(u, v_0). \end{cases} \tag{7.1}$$

The Laplace–Beltrami operator in the coordinates  $(u, v)$  has the form (see, for example, [16, p. 87])

$$\Delta = \tilde{g}^{11} \left( \frac{\partial^2}{\partial u^2} - \tilde{\Gamma}_{11}^1 \frac{\partial}{\partial u} - \tilde{\Gamma}_{11}^2 \frac{\partial}{\partial v} \right) + \tilde{g}^{22} \left( \frac{\partial^2}{\partial v^2} - \tilde{\Gamma}_{22}^1 \frac{\partial}{\partial u} - \tilde{\Gamma}_{22}^2 \frac{\partial}{\partial v} \right),$$

where the matrix  $(\tilde{g}^{ij})$  is inverse to  $(\tilde{g}_{ij})$  and  $\tilde{\Gamma}_{ij}^k$  are the Schwarz–Christoffel symbols on  $(u, v)$ .

For any function  $c = c(u) \in \mathcal{A}(\mathbb{D})$  we have

$$\Delta c = c'' \tilde{g}^{11} - c' (\tilde{g}^{11} \tilde{\Gamma}_{11}^1 + \tilde{g}^{22} \tilde{\Gamma}_{22}^1). \tag{7.2}$$

Vanishing of the second term of asymptotic in a commutator, or equivalently the condition (5.8), leads to the following theorem.

**Theorem 7.1.** *Let  $\mathcal{A}(\mathbb{D})$  be a 2-rich space of smooth functions which generates for each  $h \in (0, 1)$  the commutative  $C^*$ -algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  of Toeplitz operators. Then the common gradient lines of all real-valued functions in  $\mathcal{A}(\mathbb{D})$  are geodesics in the hyperbolic geometry of the unit disk  $\mathbb{D}$ .*

**Proof.** Given two real-valued functions  $a = a(u), b = b(u) \in \mathcal{A}(\mathbb{D})$ , condition (5.8) is equivalent to

$$\begin{aligned} 0 &\equiv a' \cdot \frac{\partial \Delta b}{\partial v} - b' \cdot \frac{\partial \Delta a}{\partial v} \\ &= (a'b'' - b'a'') \frac{\partial \tilde{g}^{11}}{\partial v} - (a'b' - b'a') \frac{\partial}{\partial v} (\tilde{g}^{11} \tilde{\Gamma}_{11}^1 + \tilde{g}^{22} \tilde{\Gamma}_{22}^1) \\ &= (a'b'' - b'a'') \frac{\partial \tilde{g}^{11}}{\partial v}. \end{aligned}$$

Note, that vanishing of  $a'b'' - b'a''$  in an open subset of  $U$  is equivalent to the property that in this subset one of the functions,  $a$  or  $b$ , is a linear combination of the other and  $e(z) \equiv 1$ , which is impossible by Lemma 6.4. By Lemma 6.4 we can change, if necessary, in different parts of  $U$  the functions  $a$  and  $b$  from  $\mathcal{A}(\mathbb{D})$  in order to have  $a'b'' - b'a'' \neq 0$ ; then

$$\frac{\partial \tilde{g}^{11}}{\partial v} = -\tilde{g}_{11}^2 \frac{\partial \tilde{g}_{11}}{\partial v} \equiv 0, \tag{7.3}$$

or

$$\frac{\partial}{\partial v} \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle \equiv 0, \tag{7.4}$$

where  $\langle X_1, X_2 \rangle = ds^2(X_1, X_2)$  is the inner product of the vector fields  $X_1$  and  $X_2$ .

Consider now any gradient line  $\gamma$  given by (7.1). The Frenet frame  $(e_1, e_2)$  of  $\gamma$  is given by

$$e_1 = \left\| \frac{\partial}{\partial u} \right\|^{-1} \frac{\partial}{\partial u}, \quad e_2 = \left\| \frac{\partial}{\partial v} \right\|^{-1} \frac{\partial}{\partial v}.$$

By [13, p. 78], the geodesic curvature  $\kappa_\gamma(u)$  of  $\gamma$  is calculated as follows:

$$\kappa_\gamma(u) = \left\| \frac{\partial}{\partial u} \right\|^{-1} \langle e_2, \nabla_{\frac{\partial}{\partial u}} e_1 \rangle.$$

Using standard properties of the connection  $\nabla$ , we have

$$\nabla_{\frac{\partial}{\partial u}} e_1 = \left( \frac{\partial}{\partial u} \left\| \frac{\partial}{\partial u} \right\|^{-1} \right) \cdot \frac{\partial}{\partial u} + \left\| \frac{\partial}{\partial u} \right\|^{-1} \cdot \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}.$$

Thus

$$\kappa_\gamma(u) = \left\| \frac{\partial}{\partial u} \right\|^{-1} \langle e_2, \nabla_{\frac{\partial}{\partial u}} e_1 \rangle = \left\| \frac{\partial}{\partial u} \right\|^{-2} \left\| \frac{\partial}{\partial v} \right\|^{-1} \left\langle \frac{\partial}{\partial v}, \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} \right\rangle. \tag{7.5}$$

By the Koszul formula (see, for example, [16, p. 61]) we have

$$\begin{aligned} 2 \left\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle &= \frac{\partial}{\partial u} \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle + \frac{\partial}{\partial u} \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right\rangle - \frac{\partial}{\partial v} \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle \\ &\quad - \left\langle \frac{\partial}{\partial u}, \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] \right\rangle + \left\langle \frac{\partial}{\partial u}, \left[ \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right] \right\rangle + \left\langle \frac{\partial}{\partial v}, \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right] \right\rangle, \end{aligned}$$

where  $[X_1, X_2]$  is the commutator of the vector fields  $X_1$  and  $X_2$ .

Taking into account that

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 0, \quad \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] = - \left[ \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right] = 0, \quad \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right] = 0,$$

we have

$$\left\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = - \frac{1}{2} \frac{\partial}{\partial v} \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle. \tag{7.6}$$

Finally, from (5.7), (7.4)–(7.6) it follows that  $\kappa_\gamma \equiv 0$ , and thus (see, for example, [13, Proposition 4.3.2]) the system of gradient lines consists of geodesics.  $\square$

### 8. Curves with constant geodesic curvature

As it was shown in the previous section, the geodesic curvature turns out to be an important invariant to study the geometry of the gradient lines. We show that it is also fundamental to understand the nature of the level lines.

To make the paper more self-contained we list here the description of the curves in  $\mathbb{D}$  whose geodesic curvature is constant and their characterization by the curvature value.

Recall that the (canonical) Euclidean or flat metric on  $\mathbb{D}$  is given by

$$(ds^2)^E = \langle \cdot, \cdot \rangle^E = dx^2 + dy^2,$$

while the hyperbolic metric is given by the expression

$$ds^2 = \frac{dx^2 + dy^2}{\pi(1 - (x^2 + y^2))^2},$$

as introduced in (2.3). Considering these two metrics on  $\mathbb{D}$ , we have two Riemannian manifolds with  $\mathbb{D}$  as its base manifold. As a corollary, we need to distinguish the usual, Euclidean, norm in  $\mathbb{C}$  and the hyperbolic norm of vectors tangent to  $\mathbb{D}$ . To do this we use  $|\cdot|$  for the former and  $\|\cdot\|$  for the later, as it was already used at the end of the previous section.

Furthermore, each one of the above manifolds has a different way to compute the acceleration of a curve in  $\mathbb{D}$  which, with the hyperbolic metric and when defined as in [16], for a curve  $\gamma$  is given by the covariant derivative  $\nabla_{\gamma'}\gamma'$ , where  $\nabla$  is the Levi-Civita or Riemannian connection of the hyperbolic metric (see [16, Theorem 11]). Similarly, the acceleration of a curve  $\gamma$  for the Euclidean geometry is given by  $\nabla_{\gamma'}^E\gamma'$ , where  $\nabla^E$  is the Levi-Civita connection of the Euclidean metric. Since the Euclidean and hyperbolic metrics have different Levi-Civita connections, the corresponding accelerations are not the same. Because of this, if  $\gamma$  is a  $C^2$  curve in  $\mathbb{D}$ , we will denote with  $\gamma''$  and  $\ddot{\gamma}$  the acceleration of  $\gamma$  for the Euclidean and hyperbolic metrics, respectively. Lemma 14 in [16] provides the Schwarz–Christoffel symbols of the Euclidean metric, which can be used to obtain the general formula for the corresponding acceleration of a curve. From this it follows that:

$$\gamma'' = (x'', y''),$$

where  $\gamma = (x, y)$  and  $x'', y''$  simply denote the second derivatives of the real-valued one-variable functions  $x, y$ . On the other hand, we have for every  $t$  in the domain of  $\gamma$ :

$$\begin{aligned} \ddot{\gamma}(t) = & (x''(t) + \Gamma_{11}^1(\gamma(t))x'(t)^2 + 2\Gamma_{12}^1(\gamma(t))x'(t)y'(t) + \Gamma_{22}^1(\gamma(t))y'(t)^2, \\ & y''(t) + \Gamma_{11}^2(\gamma(t))x'(t)^2 + 2\Gamma_{12}^2(\gamma(t))x'(t)y'(t) + \Gamma_{22}^2(\gamma(t))y'(t)^2), \end{aligned}$$

where the functions  $\Gamma_{jk}^l$  denote the Schwarz–Christoffel symbols of the hyperbolic metric on  $\mathbb{D}$  with respect to the coordinates given by the real and imaginary part of complex numbers. From now on, we will refer to these coordinates as the natural coordinates in  $\mathbb{D}$ .

The next result states that, for a curve in  $\mathbb{D}$  that passes through the origin, the Euclidean and hyperbolic geodesic curvatures at the origin differ only by a multiplicative constant which does not depend on the curve. Of course the constant would vary if we move from the origin, since the

Euclidean and hyperbolic geodesic curvatures are not the same. However, in our computations we will be dealing with curves through the origin. This can be done since the unit disk  $\mathbb{D}$  is homogeneous as a Riemannian manifold with respect to the Möbius transformations.

We denote by  $\kappa_\gamma$  and  $\kappa_\gamma^E$  the geodesic curvatures of  $\gamma$  for the hyperbolic and Euclidean metrics, respectively.

**Lemma 8.1.** *Let  $\gamma : I \rightarrow \mathbb{D}$  be a  $C^2$  curve, where  $I$  is an open interval of  $\mathbb{R}$ . Suppose that there exists  $t_0 \in I$  such that  $\gamma(t_0) = 0$ , the origin in  $\mathbb{D}$ . Then*

$$\kappa_\gamma(t_0) = \sqrt{\pi} \kappa_\gamma^E(t_0).$$

**Proof.** Using our expression for the hyperbolic metric  $g$  and Proposition 13 from [16, p. 62] we find, with respect to the natural coordinates in  $\mathbb{D}$ , the following expressions for the components of the metric and the Schwarz–Christoffel symbols:

$$g_{jk}(0) = \frac{1}{\pi} \delta_{jk}, \quad \Gamma_{jk}^l(0) = 0, \tag{8.1}$$

for every  $j, k, l = 1, 2$ . In particular at the origin of  $\mathbb{D}$ , we have

$$\| \cdot \| = \frac{1}{\sqrt{\pi}} | \cdot |. \tag{8.2}$$

Thus the formulas for the acceleration of a curve in  $\mathbb{D}$ , for both the hyperbolic and the Euclidean metrics imply

$$\gamma''(t_0) = \ddot{\gamma}(t_0). \tag{8.3}$$

On the other hand, by [13, Definition 4.2], it follows that the Frenet frame for  $\gamma$  at  $t_0$  with respect to the Euclidean metric is given by

$$e_1^E(t_0) = \frac{\gamma'(t_0)}{|\gamma'(t_0)|},$$

$$e_2^E(t_0) = \frac{(-y'(t_0), x'(t_0))}{|\gamma'(t_0)|} = \frac{i\gamma'(t_0)}{|\gamma'(t_0)|},$$

and by (8.1) the Frenet frame for  $\gamma$  at  $t_0$  with respect to the hyperbolic metric is given by

$$e_1(t_0) = \frac{\gamma'(t_0)}{\|\gamma'(t_0)\|} = \sqrt{\pi} e_1^E(t_0),$$

$$e_2(t_0) = \frac{(-y'(t_0), x'(t_0))}{\|\gamma'(t_0)\|} = \frac{i\gamma'(t_0)}{\|\gamma'(t_0)\|} = \sqrt{\pi} e_2^E(t_0).$$

Hence, the formula for the geodesic curvature considered in Section 7 (see also [13]) yields for the Euclidean metric

$$\begin{aligned} \kappa_\gamma^E(t_0) &= \frac{1}{|\gamma'(t_0)|} \left\langle e_2^E(t_0), \left( \nabla_{\gamma'(t_0)}^E \frac{1}{|\gamma'|} \gamma' \right)(t_0) \right\rangle^E \\ &= \frac{1}{|\gamma'(t_0)|^2} \left\langle i\gamma'(t_0), \frac{1}{|\gamma'(t_0)|} (\nabla_{\gamma'(t_0)}^E \gamma')(t_0) \right\rangle^E \\ &= \frac{1}{|\gamma'(t_0)|^3} \langle i\gamma'(t_0), \gamma''(t_0) \rangle^E, \end{aligned}$$

where we have used for the second identity the derivation properties of  $\nabla^E$  together with the fact that  $e_2(t_0)$  and  $\gamma'(t_0)$  are perpendicular. Correspondingly, we have for the hyperbolic metric:

$$\begin{aligned} \kappa_\gamma(t_0) &= \frac{1}{\|\gamma'(t_0)\|} \left\langle e_2(t_0), \left( \nabla_{\gamma'(t_0)} \frac{1}{\|\gamma'\|} \gamma' \right)(t_0) \right\rangle \\ &= \frac{1}{\|\gamma'(t_0)\|^2} \left\langle i\gamma'(t_0), \frac{1}{\|\gamma'(t_0)\|} (\nabla_{\gamma'(t_0)} \gamma')(t_0) \right\rangle \\ &= \frac{1}{\|\gamma'(t_0)\|^3} \langle i\gamma'(t_0), \ddot{\gamma}(t_0) \rangle \\ &= \frac{\sqrt{\pi}}{|\gamma'(t_0)|^3} \langle i\gamma'(t_0), \gamma''(t_0) \rangle^E, \end{aligned}$$

where the fourth identity follows from Eqs. (8.1)–(8.3). The result is then a consequence of the formulas for these two geodesic curvatures.  $\square$

The previous lemma allows us to obtain a formula for the geodesic curvature of curves in  $\mathbb{D}$  with the hyperbolic metric. The following formula was first stated, to the best of our knowledge, by Carathéodory [5].

**Theorem 8.2.** *Let  $\gamma : I \rightarrow \mathbb{D}$  be a  $C^2$  curve, where  $I$  is an open interval in  $\mathbb{R}$ . Then the geodesic curvature of  $\gamma$  with respect to the hyperbolic metric in  $\mathbb{D}$  is given by the expression:*

$$\kappa_\gamma = \sqrt{\pi} \left( 2 \frac{\langle i\gamma, \gamma' \rangle^E}{|\gamma'|} + \frac{(1 - |\gamma|^2) \langle i\gamma', \gamma'' \rangle^E}{|\gamma'|^3} \right).$$

**Proof.** Let  $t_0 \in I$  be given. Consider the Möbius transformation given by

$$\phi(z) = \frac{z - \gamma(t_0)}{1 - \overline{\gamma(t_0)}z},$$

which maps  $\mathbb{D}$  onto  $\mathbb{D}$  taking  $\gamma(t_0)$  into the origin. Let us denote

$$\alpha = \phi \circ \gamma.$$

From Lemma 8.1 and the fact that  $\phi$  (being a hyperbolic isometry) preserves geodesic curvatures it follows that

$$\kappa_\gamma(t_0) = \kappa_\alpha(t_0) = \sqrt{\pi} \kappa_\alpha^E(t_0). \tag{8.4}$$

On the other hand, using the arguments from the proof of Lemma 8.1 we conclude that

$$\kappa_\alpha^E = \frac{\langle i\alpha', \alpha'' \rangle^E}{|\alpha'|^3}. \tag{8.5}$$

Easy calculations show that

$$\begin{aligned} \alpha'(t_0) &= \frac{\gamma'(t_0)}{1 - |\gamma(t_0)|^2}, \\ \alpha''(t_0) &= \frac{\gamma''(t_0)(1 - |\gamma(t_0)|^2) + 2\overline{\gamma(t_0)}\gamma'(t_0)^2}{(1 - |\gamma(t_0)|^2)^2}. \end{aligned}$$

Substituting to (8.5) we have

$$\begin{aligned} \kappa_\alpha^E(t_0) &= \frac{(1 - |\gamma(t_0)|^2)^3}{|\gamma'(t_0)|^3} \left\langle \frac{i\gamma'(t_0)}{1 - |\gamma(t_0)|^2}, \frac{\gamma''(t_0)(1 - |\gamma(t_0)|^2) + 2\overline{\gamma(t_0)}\gamma'(t_0)^2}{(1 - |\gamma(t_0)|^2)^2} \right\rangle^E \\ &= \frac{1}{|\gamma'(t_0)|^3} \langle i\gamma'(t_0), \gamma''(t_0)(1 - |\gamma(t_0)|^2) + 2\overline{\gamma(t_0)}\gamma'(t_0)^2 \rangle^E \\ &= 2 \frac{\langle i\gamma(t_0), \gamma'(t_0) \rangle^E}{|\gamma'(t_0)|} + \frac{(1 - |\gamma(t_0)|^2) \langle i\gamma'(t_0), \gamma''(t_0) \rangle^E}{|\gamma'(t_0)|^3}. \end{aligned}$$

The final result now follows from (8.4).  $\square$

In what follows we will use the notion of circular arcs understanding them in the extended sense of the Riemann sphere geometry. In particular, every segment of a straight line is considered to be a circular arc.

**Theorem 8.3.** *Each circular arc contained in  $\mathbb{D}$  has constant hyperbolic geodesic curvature.*

**Proof.** Every circular arc contained in  $\mathbb{D}$  can be parameterized by either one of the following curves:

$$\begin{aligned} \gamma_1 : I_1 &\rightarrow \mathbb{D}, & \gamma_2 : I_2 &\rightarrow \mathbb{D}, \\ t &\mapsto r_1 e^{r_2 i t} + z, & t &\mapsto t w + i r_3 w, \end{aligned}$$

where  $r_1, r_2, r_3 \in \mathbb{R}$  ( $r_1, r_2 \neq 0$ ),  $z, w \in \mathbb{C}$  ( $w \neq 0$ ) and  $I_1, I_2$  are suitable open intervals in  $\mathbb{R}$ . For the expression of  $\gamma_2$  we have used the fact that we can translate the origin in the parameter  $t$  to assume that at time  $t = 0$  the curve  $\gamma_2$  passes through the point in the line segment nearest to the origin. Also observe that we require the conditions:

$$|z| + |r_1| \leq 1, \quad \text{or} \quad ||z| - |r_1|| < 1, \tag{8.6}$$

$$|r_3 w| < 1, \tag{8.7}$$

for the circular arcs to have nonempty intersection with  $\mathbb{D}$ .



Then a straightforward application of the formula from Theorem 8.2 shows that the (hyperbolic) geodesic curvatures of the circular arcs defined by  $\gamma_1$  and  $\gamma_2$  are given by

$$\kappa_{\gamma_1}(t) = \frac{\text{sign}(r_2)\sqrt{\pi}(r_1^2 - |z|^2 + 1)}{|r_1|}, \tag{8.8}$$

$$\kappa_{\gamma_2}(t) = -2\sqrt{\pi}|w|r_3, \tag{8.9}$$

for every  $t$  in  $I_1, I_2$ , respectively. In particular, such geodesic curvatures do not depend on  $t$  and thus the geodesic curvature is constant for every circular arc.  $\square$

The above formulas for the geodesic curvature of circular arcs allow us to describe the cycles in  $\mathbb{D}$  by the value of their curvature as follows:

- (1) *Horocycles.* These are given by circular arcs contained in  $\mathbb{D}$  which are tangent to the boundary of  $\mathbb{D}$ . It is easy to check that such circular arcs are obtained from  $\gamma_1$  as in the proof of Theorem 8.3 precisely when  $|z| = 1 - |r_1|$ . Hence Eq. (8.8) shows that every horocycle has (constant) geodesic curvature with value either  $2\sqrt{\pi}$  or  $-2\sqrt{\pi}$  according to whether  $r_2$  is positive or negative, respectively. It is easy to check that the sign of such curvature corresponds to the orientation of the horocycle.
- (2) *Elliptic cycles.* These correspond to circles completely contained in  $\mathbb{D}$ . Such curves are obtained from  $\gamma_1$  in the proof of Theorem 8.3 precisely when  $|z| < 1 - |r_1|$ . By using this condition and Eq. (8.8) it follows that every elliptic cycle has (constant) geodesic curvature whose value lies in the set  $(-\infty, -2\sqrt{\pi}) \cup (2\sqrt{\pi}, +\infty)$ . For each given elliptic cycle, the interval of such set in which the value of the geodesic curvature lies depends on the sign of  $r_2$ . Again, this reflects the orientation of the elliptic cycle. Also, from Eq. (8.8) it is easy to see that for every  $\kappa_0 \in (-\infty, -2\sqrt{\pi}) \cup (2\sqrt{\pi}, +\infty)$  there is an elliptic cycle whose geodesic curvature has (constant) value  $\kappa_0$ .
- (3) *Hypercycles.* These are given by curves  $\gamma_1$  precisely when  $|z| > 1 - |r_1|$  and by all curves  $\gamma_2$  as in the proof of Theorem 8.3. Such choices correspond to circular arcs that intersect the boundary of  $\mathbb{D}$  in two different points. Using Eqs. (8.8) and (8.9) it is easy to see that every hypercycle has (constant) geodesic curvature whose value lies in the interval  $(-2\sqrt{\pi}, 2\sqrt{\pi})$ . Moreover, from such equations it is easy to see that for every  $\kappa_0 \in (-2\sqrt{\pi}, 2\sqrt{\pi})$  there is a hypercycle whose geodesic curvature has (constant) value  $\kappa_0$ .

Recall that, for a 1-dimensional manifold  $C$  in the complex plane (e.g., a circular arc in  $\mathbb{D}$  or a line in the complex plane), an orientation is an equivalence class of parameterizations where two of them  $\gamma_1, \gamma_2$  are considered equivalent if there is a strictly increasing function  $h$  such that  $\gamma_2 = \gamma_1 \circ h$ . A curve is called oriented if it is endowed with an orientation. With these definitions, we say that two oriented curves  $C_1$  and  $C_2$  are tangent with the same orientation at a point  $z$  if they have parameterizations  $\gamma_1$  and  $\gamma_2$ , respectively, such that for some  $t_1, t_2$  we have  $\gamma_1(t_1) = z = \gamma_2(t_2)$  and  $\gamma_1'(t_1) = c\gamma_2'(t_2)$ , for some  $c \in (0, +\infty)$ . For a given oriented curve  $C$  we say that an oriented straight line  $L$  is the oriented tangent for  $C$  at  $z$  if both are tangent with the same orientation at  $z$ .

Given  $r \in (-1, 1)$ , a simple computation shows that all oriented circular arcs that pass through  $r = r + i0 \in \mathbb{D}$  with oriented tangent at  $r$  given by  $\mathbb{R}i$  (oriented in the direction of  $i$ ) are as follows:

$$\gamma_1(t) = (r - \mu)e^{\text{sign}(r-\mu)it} + \mu, \quad \gamma_2(t) = ti + r,$$

where  $\mu \in \mathbb{R} \setminus \{r\}$ . From Eqs. (8.8) and (8.9) we conclude that the (hyperbolic) geodesic curvatures of these curves through  $r$  are given by

$$\kappa_{\gamma_1} = \frac{\sqrt{\pi}(r^2 - 2\mu r + 1)}{r - \mu}, \quad \kappa_{\gamma_2} = 2\sqrt{\pi}r.$$

A straightforward computation shows that  $\kappa_{\gamma_1}$ , as a function of  $\mu$ , has strictly positive derivative and so it is injective. Moreover, we observe that  $\kappa_{\gamma_1}$  has limits  $+\infty$ ,  $2\sqrt{\pi}r$  and  $-\infty$  for  $\mu \rightarrow r^-$ ,  $|\mu| \rightarrow +\infty$  and  $\mu \rightarrow r^+$ , respectively. From these remarks it follows that for every  $r \in (-1, 1)$  and  $\kappa_0 \in \mathbb{R}$  there is exactly one circular arc passing through  $r$  with oriented tangent line  $\mathbb{R}i$  (oriented in the direction of  $i$ ) and (constant) geodesic curvature  $\kappa_0$ . In particular, for fixed  $r$ , no circular arc given by the curves  $\gamma_1$  has geodesic curvature equal to that of  $\gamma_2$ .

On the other hand, it is well known (both from geometry and complex analysis) that the Möbius transformations fixing  $\mathbb{D}$  and the reflections through geodesics in  $\mathbb{D}$  generate a group  $G$  that acts transitively on the (hyperbolic) unit tangent bundle of  $\mathbb{D}$ . In other words, for every  $z_1, z_2 \in \mathbb{D}$  and  $v_1 \in T_{z_1}\mathbb{D}$ ,  $v_2 \in T_{z_2}\mathbb{D}$  with  $\|v_1\| = \|v_2\|$ , there is a transformation  $\phi \in G$  that maps  $\phi(z_1) = z_2$  and  $d\phi(v_1) = v_2$ . It is also known that the transformations in  $G$  map circular arcs into circular arcs and (being hyperbolic isometries) preserve the geodesic curvature. These facts and the above remarks concerning circular arcs passing through a point  $r \in (-1, 1)$  with tangent  $\mathbb{R}i$  show that, for every  $z \in \mathbb{D}$ ,  $v \in T_z\mathbb{D} \setminus \{0\}$  and  $\kappa_0 \in \mathbb{R}$ , there is exactly one oriented circular arc passing through  $z$  with (constant) geodesic curvature  $\kappa_0$  and oriented tangent line  $\mathbb{R}v$  (with orientation given by  $v$ ).

The above observations ensure the existence of curves with constant geodesic curvature and prescribed initial conditions. We also obtained uniqueness when we restrict ourselves to the family of circular arcs in  $\mathbb{D}$ . The next result proves that the uniqueness part holds for the larger family of  $C^2$  curves in  $\mathbb{D}$ . This will play an important role in our main results.

**Theorem 8.4.** *Let  $\alpha : I \rightarrow \mathbb{D}$  and  $\beta : J \rightarrow \mathbb{D}$  be  $C^2$  curves ( $I, J \subset \mathbb{R}$  open intervals) with constant geodesic curvature such that:*

- (1)  $\kappa_\alpha \equiv \kappa_\beta$ ,
- (2) both  $\|\alpha'\|$  and  $\|\beta'\|$  are constant,
- (3)  $\alpha(t_0) = \beta(t_0)$  and  $\alpha'(t_0) = \beta'(t_0)$ ,

for some  $t_0 \in I \cap J$ . Then,  $\alpha|_{I \cap J} = \beta|_{I \cap J}$ .

**Proof.** Let  $\gamma : I \rightarrow \mathbb{D}$  be a  $C^2$  curve, where  $I \subset \mathbb{R}$ . To require  $\|\gamma'\|$  to be constant is equivalent to

$$\langle \gamma', \ddot{\gamma} \rangle = 0. \tag{8.10}$$

On the other hand, by the formula of Theorem 8.2 it follows that to require from  $\gamma$  to have constant geodesic curvature is equivalent to

$$\sqrt{\pi} \left( 2 \frac{\langle i\gamma, \gamma' \rangle^E}{|\gamma'|} + \frac{(1 - |\gamma|^2) \langle i\gamma', \gamma'' \rangle^E}{|\gamma'|^3} \right) = \kappa_0, \tag{8.11}$$

for some  $\kappa_0$ .

Hence, the curves  $\alpha$  and  $\beta$  in the statement satisfy the second order system of ordinary differential equations given by (8.10) and (8.11) with same initial conditions by (3). Then, the conclusion follows from standard results on ordinary differential equations, see, for example, [16, Section “Integral curves”].  $\square$

As a consequence of the previous result we obtain the following characterization of curves in  $\mathbb{D}$  with constant hyperbolic geodesic curvature.

**Corollary 8.5.** *Let  $\alpha : I \rightarrow \mathbb{D}$  be a  $C^2$  curve with constant hyperbolic geodesic curvature, where  $I$  is an open interval in  $\mathbb{R}$ . Then, there is a curve  $\gamma : J \rightarrow \mathbb{D}$  that parameterizes a circular arc such that  $I \subset J$  and  $\alpha = \gamma|_I$ . Moreover, no open circular arc in  $\mathbb{D}$  contains the image of  $\gamma$  as a proper subset.*

**Proof.** We observe that the discussion in [16, p. 27] implies that a system of ordinary differential equations defines a vector field whose integral curves (see [16, Definition 48, p. 27]) are precisely the solution of the system. Applying this observation to the system defined by Eqs. (8.10), (8.11) in the proof of Theorem 8.4 together with the remarks that follow Corollary 50 in [16, p. 28], we conclude that there is a unique maximal integral  $\gamma : J \rightarrow \mathbb{D}$  of the vector field associated to the system of Eqs. (8.10), (8.11). In particular,  $\gamma$  has constant hyperbolic geodesic curvature and  $\alpha = \gamma|_I$ .

On the other hand,  $\gamma$  is a curve with constant hyperbolic geodesic curvature and for its initial conditions at a given parameter (value and derivative as well as the value of its geodesic curvature) our remarks concerning the curvature of circular arcs ensure the existence of a circular arc  $C$  sharing those same initial conditions. Then the uniqueness property stated by Theorem 8.4 implies that  $\gamma$  coincides with the circular arc  $C$  in the intersection of their domains. By moving along the parameter of  $\gamma$  we conclude that  $\gamma$  can be seen as obtained from circular arcs glued together in a  $C^2$  manner, which clearly implies that  $\gamma$  is itself a circular arc. The last claim follows from the fact that  $\gamma$  contains all open circular arcs with the initial condition data that  $\gamma$  itself provides.  $\square$

Since the (open) cycles in  $\mathbb{D}$  are clearly the maximal (with respect to inclusion) circular arcs, we can sum up the previous results and remarks in the following statement.

**Theorem 8.6.** *A  $C^2$  curve contained in  $\mathbb{D}$  has constant geodesic curvature if and only if its image is a segment of a cycle.*

## 9. Third term: level lines are cycles

Let  $\mathcal{A}(\mathbb{D})$  be a linear space of  $C^3$  functions in  $\mathbb{D}$  which is 2-rich and such that  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  is commutative for each  $h \in (0, 1)$ . Given a real-valued function  $a \in \mathcal{A}(\mathbb{D})$  which is nonsingular in some open set, we keep considering the orthogonal coordinate system  $(u, v)$  introduced in Section 6. Denote with  $k_\ell$  the geodesic curvature of the level curves of  $a$  as a function of  $(u, v)$ .

Our next result provides an expression for the first derivative of the geodesic curvature  $k_\ell$  in terms of Poisson brackets.

**Theorem 9.1.** *Let  $a \in \mathcal{A}(\mathbb{D})$  be a nonconstant real-valued function, and let  $z \in \mathbb{D}$  be a nonsingular point of  $a$ . Then, in a neighborhood of  $z$ , we have*

$$\frac{\partial k_\ell}{\partial v} = \frac{g\sqrt{\tilde{g}^{11}}}{(a')^2 D} \{a, \Delta a\},$$

where  $g$ ,  $D$  and  $\tilde{g}^{11}$  are as in Section 6, and  $a'$  denotes the partial derivative of  $a$  with respect to  $u$  to emphasize the independence of  $a$  with respect to  $v$ .

**Proof.** By the definition of the geodesic curvature and our previous notation we have

$$k_\ell = \left\| \frac{\partial}{\partial v} \right\|^{-2} \left\| \frac{\partial}{\partial u} \right\|^{-1} \left\langle \nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right\rangle = \frac{\tilde{\Gamma}_{22}^1 \tilde{g}_{11}}{\tilde{g}_{22} \sqrt{\tilde{g}_{11}}} = \tilde{g}^{22} \tilde{\Gamma}_{22}^1 \sqrt{\tilde{g}_{11}},$$

from which we conclude the expression

$$\frac{\partial k_\ell}{\partial v} = \frac{\partial}{\partial v} (\tilde{g}^{22} \tilde{\Gamma}_{22}^1) \sqrt{\tilde{g}_{11}}, \tag{9.1}$$

where we used the fact that  $\tilde{g}_{11}$  does not depend on  $v$  (see (7.3)). We also observe that

$$\tilde{\Gamma}_{11}^1 = \frac{1}{2} \tilde{g}_{11} \frac{\partial \tilde{g}_{11}}{\partial u},$$

and so  $\tilde{\Gamma}_{11}^1$  does not depend on  $v$  either.

As our function  $a$  depends only on  $u$ , by (7.2) we have

$$\Delta a = a'' \tilde{g}_{11} - a' (\tilde{g}^{11} \tilde{\Gamma}_{11}^1 + \tilde{g}^{22} \tilde{\Gamma}_{22}^1).$$

Calculating the Poisson brackets in the  $(u, v)$  coordinates (see (6.1)) we obtain

$$\{a, \Delta a\} = -g^{-1} D a' \frac{\partial \Delta a}{\partial v} = g^{-1} D (a')^2 \frac{\partial}{\partial v} (\tilde{g}^{22} \tilde{\Gamma}_{22}^1), \tag{9.2}$$

and our formula follows from (9.1) and (9.2) by eliminating  $\frac{\partial}{\partial v} (\tilde{g}^{22} \tilde{\Gamma}_{22}^1)$ .  $\square$

We will need the following result that computes the dimension of the jet spaces of rich symbol sets.

**Lemma 9.2.** *Let  $\mathcal{A}(\mathbb{D})$  be a  $k$ -rich complex vector space of smooth functions, where  $k \geq 2$ , and let  $S$  be the set of common singular points of the elements in  $\mathcal{A}(\mathbb{D})$ . If  $\{a, b\} = 0$  for every  $a, b \in \mathcal{A}(\mathbb{D})$ , then the complex dimension of  $J_z^l(\mathcal{A}(\mathbb{D}))$  is  $l + 1$  at every  $z \in \mathbb{D} \setminus S$  and for every  $l = 0, \dots, k$ .*

**Proof.** First observe that, by the definition of richness,  $S$  is finite. Now choose  $z \in \mathbb{D} \setminus S$ . From the arguments in the first paragraphs of Section 6 and those found in the proof of Lemma 6.5, our hypothesis ensures the existence of a smooth real coordinate system  $(u, v)$  such that, in a neighborhood of  $z$ , every  $a \in \mathcal{A}(\mathbb{D})$  is a (complex) function of  $u$ , i.e. it is independent of  $v$ .

Fix any  $a \in \mathcal{A}(\mathbb{D})$ . From the above remarks, there is a complex valued function of one real variable such that  $a = h \circ u$ . In particular, all partial derivatives of  $a$  at  $z$  that involve at least one with respect to  $v$  vanish. In other words, the only possibly nonvanishing partial derivatives of  $a$  at  $z$  are those of the form

$$\frac{\partial^j a}{\partial u^j}(z).$$

Hence, for every  $l \geq 0$ , the  $l$ -jet at  $z$  of  $a$  is of the form

$$j_z^l(a) = \left( a(z), \frac{\partial a}{\partial u}(z), \frac{\partial^2 a}{\partial u^2}(z), \dots, \frac{\partial^l a}{\partial u^l}(z) \right).$$

Such representation clearly defines a linear inclusion of  $J_z^l(\mathcal{A}(\mathbb{D}))$  into  $\mathbb{C}^{l+1}$ , thus showing that the complex dimension of  $J_z^l(\mathcal{A}(\mathbb{D}))$  is at most  $l + 1$ . Then the result follows by the  $k$ -richness condition.  $\square$

In the proof of the previous result, it turns out that the natural linear map  $J_z^l(\mathcal{A}(\mathbb{D})) \rightarrow \mathbb{C}^{l+1}$  given by

$$j_z^l(a) \mapsto \left( a(z), \frac{\partial a}{\partial u}(z), \frac{\partial^2 a}{\partial u^2}(z), \dots, \frac{\partial^l a}{\partial u^l}(z) \right)$$

is in fact an isomorphism. Hence, if  $\mathcal{A}(\mathbb{D})$  satisfies the hypotheses of Lemma 9.2, then for every  $0 \leq l \leq k$  and  $z \in \mathbb{D} \setminus S$ , there exists some  $a \in \mathcal{A}(\mathbb{D})$  such that  $j_z^k(a) = (\delta_{jl})_{j=0, \dots, k}$ , i.e. each of its partial derivatives at  $z$  vanish up to order  $k$  except for  $\partial^l a / \partial u^l(z) = 1$ . The same sort of result holds for the real-valued functions in  $\mathcal{A}(\mathbb{D})$ . More precisely, we have the following result.

**Lemma 9.3.** *Let  $\mathcal{A}(\mathbb{D})$  and  $S$  satisfy the conditions of Lemma 9.2 and consider coordinates  $(u, v)$  as in the proof of Lemma 9.2. If  $\{a, b\} = 0$  for every  $a, b \in \mathcal{A}(\mathbb{D})$ , then at every  $z \in \mathbb{D} \setminus S$  and for every  $l = 0, \dots, k$  there is some real-valued  $a \in \mathcal{A}(\mathbb{D})$  all of whose partial derivatives with respect to  $(u, v)$  at  $z$  of order  $\leq k$  vanish except for  $\partial^l a / \partial u^l(z)$ .*

**Proof.** The complex linear isomorphism

$$J_z^k(\mathcal{A}(\mathbb{D})) \rightarrow \mathbb{C}^{k+1}$$

described above preserves conjugation and so induces an isomorphism

$$J_z^k(\mathcal{A}^{\mathbb{R}}(\mathbb{D})) \rightarrow \mathbb{R}^{k+1},$$

where  $\mathcal{A}^{\mathbb{R}}(\mathbb{D})$  is the subspace of real-valued functions in  $\mathcal{A}(\mathbb{D})$ . Then the jets that are mapped into the canonical base in  $\mathbb{R}^{k+1}$  satisfy the required conclusion.  $\square$

We now prove that the vanishing of the third term of a commutator, or equivalently condition (5.9), implies that the level lines of real-valued symbols are cycles.

**Theorem 9.4.** *Let  $\mathcal{A}(\mathbb{D})$  be a 3-rich vector space of smooth functions such that  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  is commutative for each  $h \in (0, 1)$ . Then for each point  $z \in \mathbb{D} \setminus S$  there is a real-valued function  $a \in \mathcal{A}(\mathbb{D})$  which has  $z$  as a nonsingular point and*

$$\{a, \Delta a\}(z) = 0.$$

**Proof.** Condition (5.9) states that

$$\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0, \tag{9.3}$$

for all  $a, b \in \mathcal{A}(\mathbb{D})$ .

The following identities:

$$\begin{aligned} \Delta(ab) &= a\Delta b + b\Delta a + 2\langle \text{grad } a, \text{grad } b \rangle, \\ \Delta(h \circ a) &= (h' \circ a)\Delta a + (h'' \circ a)\|\text{grad } a\|^2, \\ \{a, h \circ b\} &= (h' \circ b)\{a, b\}, \\ \{a, bc\} &= b\{a, c\} + c\{a, b\}, \\ \text{grad}(h \circ a) &= (h' \circ a)\text{grad } a, \end{aligned} \tag{9.4}$$

which hold for arbitrary (smooth enough) functions on the unit disk, are straightforward consequences of the definitions of the operators involved.

Consider any point  $z \in \mathbb{D} \setminus S$ , with  $S$  as in the definition of 3-richness for  $\mathcal{A}(\mathbb{D})$ . By the 1-richness of  $\mathcal{A}(\mathbb{D})$  there is an element  $a \in \mathcal{A}(\mathbb{D})$  that has  $z$  as a nonsingular point. Since  $\mathcal{A}(\mathbb{D})$  is closed under complex conjugation we can assume that  $a$  is real valued.

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function such that  $h \circ a \in \mathcal{A}(\mathbb{D})$ . We observe that the vanishing of the Poisson brackets  $\{a, b\} = 0$  and the arguments from Section 6 imply that every real-valued  $b \in \mathcal{A}(\mathbb{D})$  can be written, in a neighborhood of  $z$ , as  $b = h \circ a$  for some  $h$ .

By (9.3) it follows that

$$\{\Delta a, \Delta(h \circ a)\} + \{a, \Delta^2(h \circ a)\} + \{\Delta^2 a, (h \circ a)\} = 0. \tag{9.5}$$

Let us now compute each one of the three terms in the latter expression. By using the identities in (9.4) we obtain

$$\begin{aligned} \{\Delta a, \Delta(h \circ a)\} &= \{\Delta a, (h' \circ a)\Delta a\} + \{\Delta a, (h'' \circ a)\|\text{grad } a\|^2\} \\ &= \Delta a\{\Delta a, (h' \circ a)\} + \{\Delta a, (h'' \circ a)\|\text{grad } a\|^2\} \\ &= (h'' \circ a)\Delta a\{\Delta a, a\} + \{\Delta a, (h'' \circ a)\|\text{grad } a\|^2\}. \end{aligned} \tag{9.6}$$

For the second term in (9.3) we have

$$\begin{aligned} \{a, \Delta^2(h \circ a)\} &= \{a, \Delta((h' \circ a)\Delta a)\} + \{a, \Delta((h'' \circ a)\|\text{grad } a\|^2)\} \\ &= \{a, (h' \circ a)\Delta^2 a\} + \{a, \Delta a\Delta(h' \circ a)\} \\ &\quad + 2\{a, \langle \text{grad}(h' \circ a), \text{grad}(\Delta a) \rangle\} + \{a, \Delta((h'' \circ a)\|\text{grad } a\|^2)\} \end{aligned}$$

$$\begin{aligned}
 &= (h' \circ a)\{a, \Delta^2 a\} + \Delta a\{a, \Delta(h' \circ a)\} + \Delta(h' \circ a)\{a, \Delta a\} \\
 &\quad + 2\{a, (h'' \circ a)\langle \text{grad } a, \text{grad}(\Delta a) \rangle\} + \{a, \Delta((h'' \circ a)\|\text{grad } a\|^2)\} \\
 &= (h' \circ a)\{a, \Delta^2 a\} + \Delta a\{a, (h'' \circ a)\Delta a\} \\
 &\quad + \Delta a\{a, (h''' \circ a)\|\text{grad } a\|^2\} \\
 &\quad + ((h'' \circ a)\Delta a + (h''' \circ a)\|\text{grad } a\|^2)\{a, \Delta a\} \\
 &\quad + 2(h'' \circ a)\{a, \langle \text{grad } a, \text{grad}(\Delta a) \rangle\} + \{a, \Delta((h'' \circ a)\|\text{grad } a\|^2)\} \\
 &= (h' \circ a)\{a, \Delta^2 a\} + 2(h'' \circ a)\Delta a\{a, \Delta a\} \\
 &\quad + (h''' \circ a)\|\text{grad } a\|^2\{a, \Delta a\} \\
 &\quad + 2(h'' \circ a)\{a, \langle \text{grad } a, \text{grad}(\Delta a) \rangle\} + \{a, \Delta((h'' \circ a)\|\text{grad } a\|^2)\}, \tag{9.7}
 \end{aligned}$$

where we used the fact that  $\{a, \|\text{grad } a\|^2\} = 0$ .

Finally, by the third equality in (9.4), the last term in (9.3) can be written as

$$\{\Delta^2 a, (h \circ a)\} = (h' \circ a)\{\Delta^2 a, a\}. \tag{9.8}$$

By replacing (9.6)–(9.8) into (9.3) and after canceling out terms we obtain

$$\begin{aligned}
 &\{\Delta a, (h'' \circ a)\|\text{grad } a\|^2\} + \{a, \Delta((h'' \circ a)\|\text{grad } a\|^2)\} + (h'' \circ a)\Delta a\{a, \Delta a\} \\
 &\quad + (h''' \circ a)\|\text{grad } a\|^2\{a, \Delta a\} + 2(h'' \circ a)\{a, \langle \text{grad } a, \text{grad}(\Delta a) \rangle\} = 0.
 \end{aligned}$$

Let us now consider the first two terms of this equation. In a neighborhood of  $z$ , we can write  $(h'' \circ a)\|\text{grad } a\|^2 = f \circ a$  for some function  $f$ . This can be proved by writing  $\|\text{grad } a\|^2$  in coordinates and using the properties obtained in the proof of Theorem 7.1 to conclude that it only depends on  $a$  in a neighborhood of  $z$ . But then a straightforward computation using the equations in (9.4) yields

$$\begin{aligned}
 &\{\Delta a, f \circ a\} + \{a, \Delta(f \circ a)\} \\
 &= (f' \circ a)\{\Delta a, a\} + \{a, (f' \circ a)\Delta a + (f'' \circ a)\|\text{grad } a\|^2\} \\
 &= (f' \circ a)\{\Delta a, a\} + (f' \circ a)\{a, \Delta a\} + \{a, (f'' \circ a)\|\text{grad } a\|^2\} \\
 &= (f' \circ a)\{\Delta a, a\} + (f' \circ a)\{a, \Delta a\} + (f'' \circ a)\{a, \|\text{grad } a\|^2\} \\
 &= 0.
 \end{aligned}$$

After such computations we are left with the identity

$$((h'' \circ a)\Delta a + (h''' \circ a)\|\text{grad } a\|^2)\{a, \Delta a\} + 2(h'' \circ a)\{a, \langle \text{grad } a, \text{grad}(\Delta a) \rangle\} = 0. \tag{9.9}$$

Now choose a local coordinate system in a neighborhood of  $z$  such that  $a$  is one of the components of our coordinate map. Then, as remarked before, every  $b \in \mathcal{A}(\mathbb{D})$  can be written in a neighborhood of  $z$ , as  $b = h \circ a$ . On the other hand, the 3-richness of  $\mathcal{A}(\mathbb{D})$ , our hypotheses and Lemma 9.3 imply that there is some real-valued  $b \in \mathcal{A}(\mathbb{D})$  all of whose partial derivatives at  $z$

up to order 3 vanish except for  $\partial^3 a / \partial u^3(z)$ , where  $(u, v)$  is a suitable coordinate system near a neighborhood of  $z$ . If we write  $b = h \circ a$  as above, then a straightforward application of the chain rule yields

$$\begin{aligned} 0 &= \frac{\partial b}{\partial u}(z) = h' \circ a(z) \frac{\partial a}{\partial u}(z), \\ 0 &= \frac{\partial^2 b}{\partial u^2}(z) = h'' \circ a(z) \left( \frac{\partial a}{\partial u}(z) \right)^2 + h' \circ a(z) \frac{\partial^2 a}{\partial u^2}(z), \\ 0 &\neq \frac{\partial^3 b}{\partial u^3}(z) = h''' \circ a(z) \left( \frac{\partial a}{\partial u}(z) \right)^3 + 3h'' \circ a(z) \frac{\partial a}{\partial u}(z) \frac{\partial^2 a}{\partial u^2}(z) + h' \circ a(z) \frac{\partial^3 a}{\partial u^3}(z). \end{aligned}$$

Since  $\partial a / \partial u(z) \neq 0$  (because  $a$  is nonsingular at  $z$  and depends only on  $u$ ) this implies  $h'(a(z)) = h''(a(z)) = 0, h'''(a(z)) \neq 0$ . If we replace such  $h$  in the above computation, we reduce Eq. (9.9) at  $z$  to

$$\|(\text{grad } a)(z)\|^2 \{a, \Delta a\}(z) = 0.$$

To end the proof we recall that  $z$  is a nonsingular point of  $a$ .  $\square$

**Corollary 9.5.** *Let  $\mathcal{A}(\mathbb{D})$  be a 3-rich vector space of smooth functions  $\mathcal{A}(\mathbb{D})$  which generates for each  $h \in (0, 1)$  the commutative  $C^*$ -algebra  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  of Toeplitz operators. Then the common level lines of all real-valued functions in  $\mathcal{A}(\mathbb{D})$  are cycles.*

**Proof.** Recall first, that by Lemma 6.5 all functions from  $\mathcal{A}(\mathbb{D})$  have the same set of level lines. Given any level line  $\ell$ , Theorem 9.4 ensures that for each point  $z \in \mathbb{D} \setminus S$  there is a function  $a \in \mathcal{A}(\mathbb{D})$  such that  $\{a, \Delta a\}(z) = 0$ . Repeating the same arguments in a neighborhood  $V_z$  of  $z$ , each point of which remains nonsingular for  $a$ , we have that  $\{a, \Delta a\} \equiv 0$  in  $V_z$ . Thus by Theorem 9.1 the line  $\ell$  has a constant geodesic curvature, and, by Theorem 8.6, is a segment of a cycle.

On the other hand, the line  $\ell$  is the inverse image of a regular value of some function  $a \in \mathcal{A}(\mathbb{D})$ , and thus is a closed subset of  $\mathbb{D}$  (considered as the (open) hyperbolic plane without the boundary). Since the only segment of a cycle that defines a closed subset of  $\mathbb{D}$  is the cycle itself, we have that  $\ell$  is a cycle.  $\square$

### 10. Commutative Toeplitz operator algebras and pencils of geodesics

To achieve our main result, we want to understand the structure of the real-valued functions on  $\mathbb{D}$  whose gradient lines define a pencil of geodesics. The following result provides a geometric characterization of such functions.

**Theorem 10.1.** *A nonconstant  $C^3$  real-valued function  $a$  in  $\mathbb{D}$  defines a pencil if and only if the following two conditions are satisfied:*

- (i) *The gradient lines of  $a$  are geodesics.*
- (ii) *Each level line of  $a$  is a cycle.*



**Proof.** By the definition of a pencil and by Theorem 8.3 it is clear that the two above conditions are necessary for  $a$  to define a pencil.

To prove the sufficiency, assume that  $a$  satisfies the two given conditions. Let  $\ell$  be a level line of  $a$ . We know that the gradient lines of  $a$  are geodesics. This fact and the uniqueness of geodesics imply that all geodesics perpendicular to  $\ell$  are gradient lines of  $a$ . Moreover, since  $\ell$  is a closed submanifold of  $\mathbb{D}$ , for every point in  $\mathbb{D}$  there is at least one geodesic perpendicular to  $\ell$  passing through that point (see [4]). This implies that the gradient lines of  $a$  are precisely the geodesics perpendicular to  $\ell$ .

Now observe that every horocycle, elliptic cycle or hypercycle is a level curve in a pencil. To prove this for a horocycle we take the pencil defined by all horocycles with same center as the given one. For an elliptic cycle we first recall that every such cycle has a *hyperbolic center*, i.e. it is the set of points at a fixed hyperbolic distance to point in  $\mathbb{D}$ ; then we take the pencil given by elliptic cycles with the same hyperbolic center. For a hypercycle we consider the two accumulation points of the hypercycle in the boundary of  $\mathbb{D}$ , and we take the pencil defined by the hypercycles with the same pair of accumulation points.

With the above facts we complete the proof as follows. The level line  $\ell$  of  $a$ , being either a horocycle, an elliptic cycle or a hypercycle, is a level line in a pencil  $\mathcal{P}$ . Then the gradient lines of both  $a$  and  $\mathcal{P}$  are the same, since such lines are precisely the geodesics perpendicular to  $\ell$ . Also, the level lines of  $a$  and  $\mathcal{P}$  are the same since they are given as the leaves of the foliation obtained by integrating the normal bundle to the common set of gradient lines. This shows that the gradient and level lines of  $a$  define the pencil  $\mathcal{P}$ .  $\square$

Now Lemma 6.5, Theorem 7.1, Corollary 9.5, and Theorem 10.1 lead directly to the following result.

**Corollary 10.2.** *Let  $\mathcal{A}(\mathbb{D})$  be a 3-rich vector space of smooth functions such that  $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$  is commutative for each  $h \in (0, 1)$ . Then there exists a pencil  $\mathcal{P}$  of geodesics in  $\mathbb{D}$  such that all functions in  $\mathcal{A}(\mathbb{D})$  are constant on the cycles of  $\mathcal{P}$ .*

Now the main result of the paper reads as follows.

**Corollary 10.3.** *Let  $\mathcal{A}(\mathbb{D})$  be a 3-rich vector space of smooth functions. Then the following three statements are equivalent:*

- (i) *there is a pencil  $\mathcal{P}$  of geodesics in  $\mathbb{D}$  such that all functions in  $\mathcal{A}(\mathbb{D})$  are constant on the cycles of  $\mathcal{P}$ ;*
- (ii) *the  $C^*$ -algebra generated by Toeplitz operators with  $\mathcal{A}(\mathbb{D})$ -symbols is commutative on each weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$ ,  $h \in (0, 1)$  (or  $\mathcal{A}_\lambda^2(\mathbb{D})$ ,  $\lambda \in (-1, +\infty)$ );*
- (iii) *the  $C^*$ -algebra generated by Toeplitz operators with  $\mathcal{A}(\mathbb{D})$ -symbols is commutative on each weighted Bergman space  $\mathcal{A}_h^2(\mathbb{D})$  (or  $\mathcal{A}_\lambda^2(\mathbb{D})$ ) for a sequence of parameters  $h_n$  with  $\lim_{n \rightarrow \infty} h_n = 0$  (or  $\lambda_n$  with  $\lim_{n \rightarrow \infty} \lambda_n = +\infty$ ).*

## 11. Three-term asymptotic expansion of a commutator

In this section we prove Theorem 5.2.

Given two six times continuously differentiable functions  $a = a(z, \bar{z})$  and  $b = b(z, \bar{z})$ , by the composition formula (5.5) we have

$$(\tilde{a}_h \star \tilde{b}_h)(z, \bar{z}) = \left(\frac{1}{h} - 1\right) \int_{\mathbb{D}} \tilde{a}_h(z, \bar{\zeta}) \tilde{b}_h(\zeta, \bar{z}) \left(\frac{(1 - |z|^2)(1 - |\zeta|^2)}{(1 - z\bar{\zeta})(1 - \zeta\bar{z})}\right)^{1/h} d\mu(\zeta).$$

Substituting by (5.4) the expressions for  $\tilde{a}_h$  and  $\tilde{b}_h$  and changing then the order of integration we have

$$\begin{aligned} (\tilde{a}_h \star \tilde{b}_h)(z, \bar{z}) &= \left(\frac{1}{h} - 1\right)^3 \int_{\mathbb{D}} \left( \int_{\mathbb{D}} a(w, \bar{w}) \left(\frac{(1 - z\bar{\zeta})(1 - w\bar{w})}{(1 - z\bar{w})(1 - w\bar{\zeta})}\right)^{1/h} d\mu(w) \right) \\ &\quad \times \left( \int_{\mathbb{D}} b(w', \bar{w}') \left(\frac{(1 - \bar{z}\zeta)(1 - w'\bar{w}')}{(1 - \zeta\bar{w}')(1 - w'\bar{z})}\right)^{1/h} d\mu(w') \right) \\ &\quad \times \left(\frac{(1 - |z|^2)(1 - |\zeta|^2)}{(1 - z\bar{\zeta})(1 - \zeta\bar{z})}\right)^{1/h} d\mu(\zeta) \\ &= \left(\frac{1}{h} - 1\right)^2 \int_{\mathbb{D}} \int_{\mathbb{D}} a(w, \bar{w}) b(w', \bar{w}') \\ &\quad \times \left(\frac{(1 - |z|^2)(1 - |w|^2)(1 - |w'|^2)}{(1 - w\bar{w}')(1 - z\bar{w})(1 - w'\bar{z})}\right)^{1/h} d\mu(w) d\mu(w') \\ &\quad \times \left(\frac{1}{h} - 1\right) \int_{\mathbb{D}} \left(\frac{(1 - w\bar{w}')(1 - \zeta\bar{\zeta})}{(1 - w\bar{\zeta})(1 - \zeta\bar{w}')}\right)^{1/h} d\mu(\zeta). \end{aligned}$$

The last integral is equal to 1, as the Wick symbol of the identity operator (see (5.4)). Change now the variables

$$w = \frac{z - s}{1 - \bar{z}s} \quad \text{and} \quad w' = \frac{z - s'}{1 - \bar{z}s'},$$

then

$$(\tilde{a}_h \star \tilde{b}_h)(z, \bar{z}) = \left(\frac{1}{h} - 1\right)^2 \int_{\mathbb{D}} \int_{\mathbb{D}} a_1(s, \bar{s}) b_1(s', \bar{s}') \left(\frac{(1 - |s|^2)(1 - |s'|^2)}{(1 - s\bar{s}')}\right)^{1/h} d\mu(s) d\mu(s'),$$

where

$$a_1(s, \bar{s}) = a\left(\frac{z - s}{1 - \bar{z}s}, \frac{\bar{z} - \bar{s}}{1 - z\bar{s}}\right) \quad \text{and} \quad b_1(s', \bar{s}') = b\left(\frac{z - s'}{1 - \bar{z}s'}, \frac{\bar{z} - \bar{s}'}{1 - z\bar{s}'}\right).$$

Set  $H = \frac{1}{h} - 1$ , then using the formula

$$(1 - s\bar{s}')^{-(H+1)} = \sum_{j=0}^{\infty} c_j s^j \bar{s}'^j,$$

where

$$c_0 = 1, \quad c_j = \frac{(H + 1)(H + 2) \dots (H + j)}{j!}, \quad j \in \mathbb{N},$$

we have (substituting  $s'$  for  $s$  in the second integral)

$$\begin{aligned} (\tilde{a}_h \star \tilde{b}_h)(z, \bar{z}) &= \sum_{j=0}^{\infty} c_j \left( \frac{H}{\pi} \int_{\mathbb{D}} a_1(s, \bar{s})(1 - |s|^2)^{H-1} s^j dv(s) \right) \\ &\quad \times \left( \frac{H}{\pi} \int_{\mathbb{D}} b_1(s, \bar{s})(1 - |s|^2)^{H-1} \bar{s}^j dv(s) \right). \end{aligned} \tag{11.1}$$

Represent the function  $a_1(s, \bar{s})$  as follows:

$$\begin{aligned} a_1(s, \bar{s}) &= a_1(0, 0) + s \frac{\partial a_1}{\partial s}(0, 0) + \bar{s} \frac{\partial a_1}{\partial \bar{s}}(0, 0) \\ &\quad + \sum_{m=2}^6 \sum_{k+l=m} \frac{s^k \bar{s}^l}{k!l!} \frac{\partial^m a_1}{\partial s^k \partial \bar{s}^l}(0, 0) + o(|s|^6 + |\bar{s}|^6), \end{aligned}$$

and do the same for the function  $b_1(s, \bar{s})$ .

Substitute the above expressions for  $a_1$  and  $b_1$  into (11.1) and calculate consequently for  $j = 0, j = 1, j = 2$ , and  $j = 3$  integrals which give nonzero contribution to the first four terms of asymptotic for  $(\tilde{a}_h \star \tilde{b}_h)(z, \bar{z})$ .

Note first that

$$\begin{aligned} \frac{H}{\pi} \int_{\mathbb{D}} (1 - |s|^2)^{H-1} s^l \bar{s}^k ds &= \frac{H}{\pi} \int_0^1 (1 - r^2)^{H-1} r^{l+k+1} dr \int_{-\pi}^{\pi} e^{i(l-k)\theta} d\theta \\ &= \begin{cases} 2H \int_0^1 (1 - r^2)^{H-1} r^{2k+1} dr, & l = k, \\ 0, & l \neq k \end{cases} \\ &= \begin{cases} H \int_0^1 (1 - u)^{H-1} u^k du, & l = k, \\ 0, & l \neq k \end{cases} \\ &= \begin{cases} H \frac{\Gamma(H)\Gamma(k+1)}{\Gamma(H+k+1)}, & l = k, \\ 0, & l \neq k \end{cases} \\ &= \begin{cases} \frac{k!}{(H+1)(H+2)\dots(H+k)}, & l = k, \\ 0, & l \neq k. \end{cases} \end{aligned}$$

For  $j = 0$  we have

$$\begin{aligned} \frac{H}{\pi} \int_{\mathbb{D}} a_1(0, 0)(1 - |s|^2)^{H-1} ds &= a_1(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} b_1(0, 0)(1 - |s|^2)^{H-1} ds &= b_1(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^2 a_1}{\partial s \partial \bar{s}}(0, 0)(1 - |s|^2)^{H-1} s \bar{s} ds &= \frac{1}{H + 1} \frac{\partial^2 a_1}{\partial s \partial \bar{s}}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^2 b_1}{\partial s \partial \bar{s}}(0, 0)(1 - |s|^2)^{H-1} s \bar{s} ds &= \frac{1}{H + 1} \frac{\partial^2 b_1}{\partial s \partial \bar{s}}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^4 a_1}{\partial s^2 \partial \bar{s}^2}(0, 0)(1 - |s|^2)^{H-1} s^2 \bar{s}^2 ds &= \frac{2}{(H + 1)(H + 2)} \frac{\partial^4 a_1}{\partial s^2 \partial \bar{s}^2}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^4 b_1}{\partial s^2 \partial \bar{s}^2}(0, 0)(1 - |s|^2)^{H-1} s^2 \bar{s}^2 ds &= \frac{2}{(H + 1)(H + 2)} \frac{\partial^4 b_1}{\partial s^2 \partial \bar{s}^2}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^6 a_1}{\partial s^3 \partial \bar{s}^3}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^6 a_1}{\partial s^3 \partial \bar{s}^3}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^6 b_1}{\partial s^3 \partial \bar{s}^3}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^6 b_1}{\partial s^3 \partial \bar{s}^3}(0, 0). \end{aligned}$$

Thus for  $j = 0$  we have the following contribution to the asymptotic:

$$\begin{aligned} (\tilde{a}_h \star \tilde{b}_h)_0(z, \bar{z}) &= a_1(0, 0)b_1(0, 0) + h \left( a_1(0, 0) \frac{\partial^2 b_1}{\partial s \partial \bar{s}}(0, 0) + b_1(0, 0) \frac{\partial^2 a_1}{\partial s \partial \bar{s}}(0, 0) \right) \\ &\quad + h^2 \frac{\partial^2 a_1}{\partial s \partial \bar{s}}(0, 0) \frac{\partial^2 b_1}{\partial s \partial \bar{s}}(0, 0) \\ &\quad + \frac{h^2}{2} \left( a_1(0, 0) \frac{\partial^4 b_1}{\partial s^2 \partial \bar{s}^2}(0, 0) + b_1(0, 0) \frac{\partial^4 a_1}{\partial s^2 \partial \bar{s}^2}(0, 0) \right) \\ &\quad + \frac{h^3}{2} \left( \frac{\partial^2 a_1}{\partial s \partial \bar{s}}(0, 0) \frac{\partial^4 b_1}{\partial s^2 \partial \bar{s}^2}(0, 0) + \frac{\partial^2 b_1}{\partial s \partial \bar{s}}(0, 0) \frac{\partial^4 a_1}{\partial s^2 \partial \bar{s}^2}(0, 0) \right) \\ &\quad + \frac{h^3}{6} \left( a_1(0, 0) \frac{\partial^6 b_1}{\partial s^3 \partial \bar{s}^3}(0, 0) + b_1(0, 0) \frac{\partial^6 a_1}{\partial s^3 \partial \bar{s}^3}(0, 0) \right) + o(h^3). \end{aligned}$$

Observe that

$$(\tilde{a}_h \star \tilde{b}_h)_0(z, \bar{z}) - (\tilde{b}_h \star \tilde{a}_h)_0(z, \bar{z}) = 0,$$

and thus for  $j = 0$  we have no contribution to the first three terms of asymptotic of the commutator  $(\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z})$ .

For  $j = 1$  the nonzero integrals are as follows:

$$\begin{aligned} \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial a_1}{\partial \bar{s}}(0, 0)(1 - |s|^2)^{H-1} s \bar{s} ds &= \frac{1}{H + 1} \frac{\partial a_1}{\partial \bar{s}}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial b_1}{\partial s}(0, 0)(1 - |s|^2)^{H-1} s \bar{s} ds &= \frac{1}{H + 1} \frac{\partial b_1}{\partial s}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^3 a_1}{\partial s \partial \bar{s}^2}(0, 0)(1 - |s|^2)^{H-1} s^2 \bar{s}^2 ds &= \frac{2}{(H + 1)(H + 2)} \frac{\partial^3 a_1}{\partial s \partial \bar{s}^2}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^3 b_1}{\partial s^2 \partial \bar{s}}(0, 0)(1 - |s|^2)^{H-1} s^2 \bar{s}^2 ds &= \frac{2}{(H + 1)(H + 2)} \frac{\partial^3 b_1}{\partial s^2 \partial \bar{s}}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^5 a_1}{\partial s^2 \partial \bar{s}^3}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^5 a_1}{\partial s^2 \partial \bar{s}^3}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^5 b_1}{\partial s^3 \partial \bar{s}^2}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^5 b_1}{\partial s^3 \partial \bar{s}^2}(0, 0). \end{aligned}$$

Taking into account that  $c_1 = H + 1$  we have

$$\begin{aligned} &(\tilde{a}_h \star \tilde{b}_h)_1(z, \bar{z}) \\ &= h \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial b_1}{\partial s}(0, 0) \\ &\quad + h^2 \left( \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial^3 b_1}{\partial \bar{s} \partial s^2}(0, 0) + \frac{\partial b_1}{\partial s}(0, 0) \frac{\partial^3 a_1}{\partial \bar{s}^2 \partial s}(0, 0) \right) + h^3 \frac{\partial^3 a_1}{\partial s \partial \bar{s}^2}(0, 0) \frac{\partial^3 b_1}{\partial s^2 \partial \bar{s}}(0, 0) \\ &\quad + \frac{h^3}{2} \left( \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial^5 b_1}{\partial s^3 \partial \bar{s}^2}(0, 0) + \frac{\partial b_1}{\partial s}(0, 0) \frac{\partial^5 a_1}{\partial s^2 \partial \bar{s}^3}(0, 0) \right) + o(h^3). \end{aligned} \tag{11.2}$$

Consider now the case  $j = 2$ :

$$\begin{aligned} \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0)(1 - |s|^2)^{H-1} s^2 \bar{s}^2 ds &= \frac{2}{(H + 1)(H + 2)} \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^2 b_1}{\partial s^2}(0, 0)(1 - |s|^2)^{H-1} s^2 \bar{s}^2 ds &= \frac{2}{(H + 1)(H + 2)} \frac{\partial^2 b_1}{\partial s^2}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^4 a_1}{\partial s \partial \bar{s}^3}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^4 a_1}{\partial s \partial \bar{s}^3}(0, 0), \end{aligned}$$

$$\frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^4 b_1}{\partial s^3 \partial \bar{s}}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds = \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^4 b_1}{\partial s^3 \partial \bar{s}}(0, 0).$$

Recall that  $c_2 = \frac{1}{2}(H + 1)(H + 2)$ , thus

$$\begin{aligned} (\tilde{a}_h \star \tilde{b}_h)_2(z, \bar{z}) &= \frac{h^2}{2} \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0) \frac{\partial^2 b_1}{\partial s^2}(0, 0) \\ &\quad + \frac{h^3}{2} \left( \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0) \frac{\partial^4 b_1}{\partial s^3 \partial \bar{s}}(0, 0) + \frac{\partial^2 b_1}{\partial s^2}(0, 0) \frac{\partial^4 a_1}{\partial s \partial \bar{s}^3}(0, 0) \right) + o(h^3). \end{aligned} \tag{11.3}$$

Consider finally the case  $j = 3$ :

$$\begin{aligned} \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^3 a_1}{\partial \bar{s}^3}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^3 a_1}{\partial \bar{s}^3}(0, 0), \\ \frac{H}{\pi} \int_{\mathbb{D}} \frac{\partial^3 b_1}{\partial s^3}(0, 0)(1 - |s|^2)^{H-1} s^3 \bar{s}^3 ds &= \frac{3!}{(H + 1)(H + 2)(H + 3)} \frac{\partial^3 b_1}{\partial s^3}(0, 0). \end{aligned}$$

Taking into account that  $c_3 = (H + 1)(H + 2)(H + 3)/3!$  we have

$$(\tilde{a}_h \star \tilde{b}_h)_3(z, \bar{z}) = \frac{h^3}{6} \frac{\partial^3 a_1}{\partial \bar{s}^3}(0, 0) \frac{\partial^3 b_1}{\partial s^3}(0, 0) + o(h^3). \tag{11.4}$$

Now from (11.2)–(11.4) we have

$$\begin{aligned} &(\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) \\ &= h \left( \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial b_1}{\partial s}(0, 0) - \frac{\partial a_1}{\partial s}(0, 0) \frac{\partial b_1}{\partial \bar{s}}(0, 0) \right) \\ &\quad + h^2 \left[ \left( \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial^3 b_1}{\partial \bar{s} \partial s^2}(0, 0) - \frac{\partial a_1}{\partial s}(0, 0) \frac{\partial^3 b_1}{\partial \bar{s}^2 \partial s}(0, 0) \right) \right. \\ &\quad + \left( \frac{\partial b_1}{\partial s}(0, 0) \frac{\partial^3 a_1}{\partial \bar{s}^2 \partial s}(0, 0) - \frac{\partial b_1}{\partial \bar{s}}(0, 0) \frac{\partial^3 a_1}{\partial \bar{s} \partial s^2}(0, 0) \right) \\ &\quad + \left. \frac{1}{2} \left( \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0) \frac{\partial^2 b_1}{\partial s^2}(0, 0) - \frac{\partial^2 a_1}{\partial s^2}(0, 0) \frac{\partial^2 b_1}{\partial \bar{s}^2}(0, 0) \right) \right] \\ &\quad + h^3 \left\{ \left( \frac{\partial^3 a_1}{\partial s \partial \bar{s}^2}(0, 0) \frac{\partial^3 b_1}{\partial s^2 \partial \bar{s}}(0, 0) - \frac{\partial^3 a_1}{\partial s^2 \partial \bar{s}}(0, 0) \frac{\partial^3 b_1}{\partial s \partial \bar{s}^2}(0, 0) \right) \right. \\ &\quad + \frac{1}{6} \left( \frac{\partial^3 a_1}{\partial \bar{s}^3}(0, 0) \frac{\partial^3 b_1}{\partial s^3}(0, 0) - \frac{\partial^3 a_1}{\partial s^3}(0, 0) \frac{\partial^3 b_1}{\partial \bar{s}^3}(0, 0) \right) \\ &\quad + \left. \frac{1}{2} \left[ \left( \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial^5 b_1}{\partial s^3 \partial \bar{s}^2}(0, 0) - \frac{\partial a_1}{\partial s}(0, 0) \frac{\partial^5 b_1}{\partial s^2 \partial \bar{s}^3}(0, 0) \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \left( \frac{\partial b_1}{\partial s}(0, 0) \frac{\partial^5 a_1}{\partial s^2 \partial \bar{s}^3}(0, 0) - \frac{\partial b_1}{\partial \bar{s}}(0, 0) \frac{\partial^5 a_1}{\partial s^3 \partial \bar{s}^2}(0, 0) \right) \Bigg] \\
 & + \frac{1}{2} \left[ \left( \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0) \frac{\partial^4 b_1}{\partial s^3 \partial \bar{s}}(0, 0) - \frac{\partial^2 a_1}{\partial s^2}(0, 0) \frac{\partial^4 b_1}{\partial s \partial \bar{s}^3}(0, 0) \right) \right. \\
 & + \left( \frac{\partial^2 b_1}{\partial s^2}(0, 0) \frac{\partial^4 a_1}{\partial s \partial \bar{s}^3}(0, 0) - \frac{\partial^2 b_1}{\partial \bar{s}^2}(0, 0) \frac{\partial^4 a_1}{\partial s^3 \partial \bar{s}}(0, 0) \right) \Bigg] \\
 & - \left[ \left( \frac{\partial a_1}{\partial \bar{s}}(0, 0) \frac{\partial^3 b_1}{\partial \bar{s} \partial s^2}(0, 0) - \frac{\partial a_1}{\partial s}(0, 0) \frac{\partial^3 b_1}{\partial \bar{s}^2 \partial s}(0, 0) \right) \right. \\
 & + \left( \frac{\partial b_1}{\partial s}(0, 0) \frac{\partial^3 a_1}{\partial \bar{s}^2 \partial s}(0, 0) - \frac{\partial b_1}{\partial \bar{s}}(0, 0) \frac{\partial^3 a_1}{\partial \bar{s} \partial s^2}(0, 0) \right) \\
 & \left. + \frac{1}{2} \left( \frac{\partial^2 a_1}{\partial \bar{s}^2}(0, 0) \frac{\partial^2 b_1}{\partial s^2}(0, 0) - \frac{\partial^2 a_1}{\partial s^2}(0, 0) \frac{\partial^2 b_1}{\partial \bar{s}^2}(0, 0) \right) \right] \Bigg\} + o(h^3). \tag{11.5}
 \end{aligned}$$

We note that the last group of terms in  $h^3$ , starting with the sign ‘-’, is originated from the group in  $h^2$  of (11.2) and (11.3) because of

$$\frac{1}{(H + 1)(H + 2)} = \frac{h^2}{1 + h} = h^2 - h^3 + o(h^3).$$

For the function

$$c_1(s, \bar{s}) = c \left( \frac{z - s}{1 - \bar{z}s}, \frac{\bar{z} - \bar{s}}{1 - z\bar{s}} \right)$$

calculate

$$\begin{aligned}
 \frac{\partial c_1}{\partial s} &= \frac{\partial c}{\partial w} \cdot (-1) \frac{1 - z\bar{z}}{(1 - \bar{z}s)^2}, \\
 \frac{\partial c_1}{\partial \bar{s}} &= \frac{\partial c}{\partial \bar{w}} \cdot (-1) \frac{1 - z\bar{z}}{(1 - z\bar{s})^2}, \\
 \frac{\partial^2 c_1}{\partial s^2} &= \frac{\partial^2 c}{\partial w^2} \cdot \frac{(1 - z\bar{z})^2}{(1 - \bar{z}s)^4} - 2\bar{z} \frac{1 - z\bar{z}}{(1 - \bar{z}s)^3} \cdot \frac{\partial c}{\partial w}, \\
 \frac{\partial^2 c_1}{\partial \bar{s}^2} &= \frac{\partial^2 c}{\partial \bar{w}^2} \cdot \frac{(1 - z\bar{z})^2}{(1 - z\bar{s})^4} - 2z \frac{1 - z\bar{z}}{(1 - z\bar{s})^3} \cdot \frac{\partial c}{\partial \bar{w}}, \\
 \frac{\partial^3 c_1}{\partial \bar{s} \partial s^2} &= -\frac{\partial^3 c}{\partial \bar{w} \partial w^2} \cdot \frac{(1 - z\bar{z})^3}{(1 - \bar{z}s)^4 (1 - z\bar{s})^2} + 2\bar{z} \frac{(1 - z\bar{z})^2}{(1 - \bar{z}s)^3 (1 - z\bar{s})^2} \cdot \frac{\partial^2 c}{\partial \bar{w} \partial w}, \\
 \frac{\partial^3 c_1}{\partial \bar{s}^2 \partial s} &= -\frac{\partial^3 c}{\partial \bar{w}^2 \partial w} \cdot \frac{(1 - z\bar{z})^3}{(1 - \bar{z}s)^2 (1 - z\bar{s})^4} + 2z \frac{(1 - z\bar{z})^2}{(1 - \bar{z}s)^2 (1 - z\bar{s})^3} \cdot \frac{\partial^2 c}{\partial \bar{w} \partial w}, \\
 \frac{\partial^3 c_1}{\partial s^3} &= -\frac{\partial^3 c}{\partial w^3} \cdot \frac{(1 - z\bar{z})^3}{(1 - \bar{z}s)^6} + 6\bar{z} \frac{(1 - z\bar{z})^2}{(1 - \bar{z}s)^5} \cdot \frac{\partial^2 c}{\partial w^2} - 6\bar{z}^2 \frac{1 - z\bar{z}}{(1 - \bar{z}s)^4} \cdot \frac{\partial c}{\partial w},
 \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 c_1}{\partial \bar{s}^3} &= -\frac{\partial^3 c}{\partial \bar{w}^3} \cdot \frac{(1-z\bar{z})^3}{(1-z\bar{s})^6} + 6z \frac{(1-z\bar{z})^2}{(1-z\bar{s})^5} \cdot \frac{\partial^2 c}{\partial \bar{w}^2} - 6z^2 \frac{1-z\bar{z}}{(1-z\bar{s})^4} \cdot \frac{\partial c}{\partial \bar{w}}, \\ \frac{\partial^4 c_1}{\partial \bar{s} \partial s^3} &= \frac{\partial^4 c}{\partial \bar{w} \partial w^3} \cdot \frac{(1-z\bar{z})^4}{(1-\bar{z}s)^6(1-z\bar{s})^2} - 6\bar{z} \frac{(1-z\bar{z})^3}{(1-\bar{z}s)^5(1-z\bar{s})^2} \cdot \frac{\partial^3 c}{\partial \bar{w} \partial w^2} \\ &\quad + 6\bar{z}^2 \frac{(1-z\bar{z})^2}{(1-\bar{z}s)^4(1-z\bar{s})^2} \cdot \frac{\partial^2 c}{\partial \bar{w} \partial w}, \\ \frac{\partial^4 c_1}{\partial \bar{s}^3 \partial s} &= \frac{\partial^4 c}{\partial \bar{w}^3 \partial w} \cdot \frac{(1-z\bar{z})^4}{(1-\bar{z}s)^2(1-z\bar{s})^6} - 6z \frac{(1-z\bar{z})^3}{(1-\bar{z}s)^2(1-z\bar{s})^5} \cdot \frac{\partial^3 c}{\partial \bar{w}^2 \partial w} \\ &\quad + 6z^2 \frac{(1-z\bar{z})^2}{(1-\bar{z}s)^2(1-z\bar{s})^4} \cdot \frac{\partial^2 c}{\partial \bar{w} \partial w}, \\ \frac{\partial^5 c_1}{\partial \bar{s}^2 \partial s^3} &= -\frac{\partial^5 c}{\partial \bar{w}^2 \partial w^3} \cdot \frac{(1-z\bar{z})^5}{(1-\bar{z}s)^6(1-z\bar{s})^4} + \frac{\partial^4 c}{\partial \bar{w} \partial w^3} \cdot \frac{2z(1-z\bar{z})^4}{(1-\bar{z}s)^6(1-z\bar{s})^3} \\ &\quad + 6\bar{z} \frac{(1-z\bar{z})^4}{(1-\bar{z}s)^5(1-z\bar{s})^4} \cdot \frac{\partial^4 c}{\partial \bar{w}^2 \partial w^2} - 12z\bar{z} \frac{(1-z\bar{z})^3}{(1-\bar{z}s)^5(1-z\bar{s})^3} \cdot \frac{\partial^3 c}{\partial \bar{w} \partial w^2} \\ &\quad - 6\bar{z}^2 \frac{(1-z\bar{z})^3}{(1-\bar{z}s)^4(1-z\bar{s})^4} \cdot \frac{\partial^3 c}{\partial \bar{w}^2 \partial w} + 12z\bar{z}^2 \frac{(1-z\bar{z})^2}{(1-\bar{z}s)^4(1-z\bar{s})^3} \cdot \frac{\partial^2 c}{\partial \bar{w} \partial w}, \\ \frac{\partial^5 c_1}{\partial \bar{s}^3 \partial s^2} &= -\frac{\partial^5 c}{\partial \bar{w}^3 \partial w^2} \cdot \frac{(1-z\bar{z})^5}{(1-\bar{z}s)^4(1-z\bar{s})^6} + \frac{\partial^4 c}{\partial \bar{w}^3 \partial w} \cdot \frac{2\bar{z}(1-z\bar{z})^4}{(1-\bar{z}s)^3(1-z\bar{s})^6} \\ &\quad + 6z \frac{(1-z\bar{z})^4}{(1-\bar{z}s)^4(1-z\bar{s})^5} \cdot \frac{\partial^4 c}{\partial \bar{w}^2 \partial w^2} - 12z\bar{z} \frac{(1-z\bar{z})^3}{(1-\bar{z}s)^3(1-z\bar{s})^5} \cdot \frac{\partial^3 c}{\partial \bar{w}^2 \partial w} \\ &\quad - 6z^2 \frac{(1-z\bar{z})^3}{(1-\bar{z}s)^4(1-z\bar{s})^4} \cdot \frac{\partial^3 c}{\partial \bar{w} \partial w^2} + 12z^2\bar{z} \frac{(1-z\bar{z})^2}{(1-\bar{z}s)^3(1-z\bar{s})^4} \cdot \frac{\partial^2 c}{\partial \bar{w} \partial w}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial c_1}{\partial s}(0,0) &= -(1-z\bar{z}) \frac{\partial c}{\partial z}(z,\bar{z}), \\ \frac{\partial c_1}{\partial \bar{s}}(0,0) &= -(1-z\bar{z}) \frac{\partial c}{\partial \bar{z}}(z,\bar{z}), \\ \frac{\partial^2 c_1}{\partial s^2}(0,0) &= (1-z\bar{z})^2 \frac{\partial^2 c}{\partial z^2}(z,\bar{z}) - 2\bar{z}(1-z\bar{z}) \frac{\partial c}{\partial z}(z,\bar{z}), \\ \frac{\partial^2 c_1}{\partial \bar{s}^2}(0,0) &= (1-z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z}^2}(z,\bar{z}) - 2z(1-z\bar{z}) \frac{\partial c}{\partial \bar{z}}(z,\bar{z}), \\ \frac{\partial^3 c_1}{\partial \bar{s} \partial s^2}(0,0) &= -(1-z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z} \partial z^2}(z,\bar{z}) + 2\bar{z}(1-z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z} \partial z}(z,\bar{z}), \\ \frac{\partial^3 c_1}{\partial \bar{s}^2 \partial s}(0,0) &= -(1-z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z}^2 \partial z}(z,\bar{z}) + 2z(1-z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z} \partial z}(z,\bar{z}), \\ \frac{\partial^3 c_1}{\partial s^3}(0,0) &= -(1-z\bar{z})^3 \frac{\partial^3 c}{\partial z^3} + 6\bar{z}(1-z\bar{z})^2 \frac{\partial^2 c}{\partial z^2} - 6\bar{z}^2(1-z\bar{z}) \frac{\partial c}{\partial z}, \end{aligned}$$



$$\begin{aligned} \frac{\partial^3 c_1}{\partial \bar{s}^3}(0, 0) &= -(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z}^3} + 6z(1 - z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z}^2} - 6z^2(1 - z\bar{z}) \frac{\partial c}{\partial \bar{z}}, \\ \frac{\partial^4 c_1}{\partial \bar{s} \partial s^3}(0, 0) &= (1 - z\bar{z})^4 \frac{\partial^4 c}{\partial \bar{z} \partial z^3} - 6\bar{z}(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z} \partial z^2} + 6\bar{z}^2(1 - z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z} \partial z}, \\ \frac{\partial^4 c_1}{\partial \bar{s}^3 \partial s}(0, 0) &= (1 - z\bar{z})^4 \frac{\partial^4 c}{\partial \bar{z}^3 \partial z} - 6z(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z}^2 \partial z} + 6z^2(1 - z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z} \partial z}, \\ \frac{\partial^5 c_1}{\partial \bar{s}^2 \partial s^3}(0, 0) &= -(1 - z\bar{z})^5 \frac{\partial^5 c}{\partial \bar{z}^2 \partial z^3} + 2z(1 - z\bar{z})^4 \frac{\partial^4 c}{\partial \bar{z} \partial z^3} + 6\bar{z}(1 - z\bar{z})^4 \frac{\partial^4 c}{\partial \bar{z}^2 \partial z^2} \\ &\quad - 12z\bar{z}(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z} \partial z^2} - 6\bar{z}^2(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z}^2 \partial z} + 12z\bar{z}^2(1 - z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z} \partial z}, \\ \frac{\partial^5 c_1}{\partial \bar{s}^3 \partial s^2}(0, 0) &= -(1 - z\bar{z})^5 \frac{\partial^5 c}{\partial \bar{z}^3 \partial z^2} + 2\bar{z}(1 - z\bar{z})^4 \frac{\partial^4 c}{\partial \bar{z}^3 \partial z} + 6z(1 - z\bar{z})^4 \frac{\partial^4 c}{\partial \bar{z}^2 \partial z^2} \\ &\quad - 12z\bar{z}(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z}^2 \partial z} - 6z^2(1 - z\bar{z})^3 \frac{\partial^3 c}{\partial \bar{z} \partial z^2} + 12z^2\bar{z}(1 - z\bar{z})^2 \frac{\partial^2 c}{\partial \bar{z} \partial z}. \end{aligned}$$

We start with the two-term asymptotic expansion of the commutator. Substituting the first six expressions of the above derivatives to (11.5) we have

$$\begin{aligned} (\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) &= h(1 - z\bar{z})^2 \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} \right) \\ &\quad + h^2 \left[ (1 - z\bar{z})^4 \left[ \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^3 b}{\partial \bar{z} \partial z^2} - \frac{\partial a}{\partial z} \frac{\partial^3 b}{\partial \bar{z}^2 \partial z} \right) \right. \right. \\ &\quad \left. \left. + \left( \frac{\partial b}{\partial z} \frac{\partial^3 a}{\partial \bar{z}^2 \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^3 a}{\partial \bar{z} \partial z^2} \right) + \frac{1}{2} \left( \frac{\partial^2 a}{\partial \bar{z}^2} \frac{\partial^2 b}{\partial z^2} - \frac{\partial^2 a}{\partial z^2} \frac{\partial^2 b}{\partial \bar{z}^2} \right) \right] \right. \\ &\quad \left. - 2(1 - z\bar{z})^3 \left[ \bar{z} \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial \bar{z} \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial \bar{z} \partial z} \right) + z \left( \frac{\partial b}{\partial z} \frac{\partial^2 a}{\partial \bar{z} \partial z} - \frac{\partial a}{\partial z} \frac{\partial^2 b}{\partial \bar{z} \partial z} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left( \bar{z} \left( \frac{\partial b}{\partial z} \frac{\partial^2 a}{\partial \bar{z}^2} - \frac{\partial a}{\partial z} \frac{\partial^2 b}{\partial \bar{z}^2} \right) + z \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial z^2} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial z^2} \right) \right) \right] \right. \\ &\quad \left. + 2z\bar{z}(1 - z\bar{z})^2 \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} \right) \right] + o(h^2). \end{aligned}$$

Recall (see (5.2) and (5.3)), that the Poisson bracket on the unit disk is given by

$$\{a, b\} = 2\pi i(1 - z\bar{z})^2 \left( \frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} - \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} \right),$$

and that the Laplace–Beltrami operator has the form

$$\Delta = 4\pi(1 - z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

Observe now that

$$\begin{aligned}
\frac{i}{8\pi^2} \Delta\{a, b\} &= (1 - z\bar{z})^2 \frac{\partial}{\partial z} \left( \frac{\partial}{\partial \bar{z}} (1 - z\bar{z})^2 \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial a}{\partial z} \right) \right) \\
&= (1 - z\bar{z})^2 \frac{\partial}{\partial z} \left( -2z(1 - z\bar{z}) \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial a}{\partial z} \right) \right. \\
&\quad \left. + (1 - z\bar{z})^2 \left( \frac{\partial^2 a}{\partial \bar{z}^2} \frac{\partial b}{\partial z} + \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial \bar{z} \partial z} - \frac{\partial^2 b}{\partial \bar{z}^2} \frac{\partial a}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial \bar{z} \partial z} \right) \right) \\
&= (1 - z\bar{z})^2 \left( -2(1 - z\bar{z}) \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial a}{\partial z} \right) + 2z\bar{z} \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial a}{\partial z} \right) \right. \\
&\quad \left. - 2z(1 - z\bar{z}) \left( \frac{\partial^2 a}{\partial \bar{z} \partial z} \frac{\partial b}{\partial z} + \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial z^2} - \frac{\partial^2 b}{\partial \bar{z} \partial z} \frac{\partial a}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial z^2} \right) \right. \\
&\quad \left. - 2\bar{z}(1 - z\bar{z}) \left( \frac{\partial^2 a}{\partial \bar{z}^2} \frac{\partial b}{\partial z} + \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial \bar{z} \partial z} - \frac{\partial^2 b}{\partial \bar{z}^2} \frac{\partial a}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial \bar{z} \partial z} \right) \right. \\
&\quad \left. + (1 - z\bar{z})^2 \left( \frac{\partial^3 a}{\partial \bar{z}^2 \partial z} \frac{\partial b}{\partial z} + \frac{\partial^2 a}{\partial \bar{z}^2} \frac{\partial^2 b}{\partial z^2} + \frac{\partial^2 a}{\partial \bar{z} \partial \bar{z}} \frac{\partial^2 b}{\partial \bar{z} \partial z} + \frac{\partial a}{\partial \bar{z}} \frac{\partial^3 b}{\partial \bar{z} \partial z^2} \right. \right. \\
&\quad \left. \left. - \frac{\partial^3 b}{\partial \bar{z}^2 \partial z} \frac{\partial a}{\partial z} - \frac{\partial^2 b}{\partial \bar{z}^2} \frac{\partial^2 a}{\partial z^2} - \frac{\partial^2 b}{\partial \bar{z} \partial z} \frac{\partial^2 a}{\partial \bar{z} \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^3 a}{\partial \bar{z} \partial z^2} \right) \right) \\
&= (1 - z\bar{z})^4 \left[ \left( \frac{\partial^2 a}{\partial \bar{z}^2} \frac{\partial^2 b}{\partial z^2} - \frac{\partial^2 b}{\partial \bar{z}^2} \frac{\partial^2 a}{\partial z^2} \right) \right. \\
&\quad \left. + \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^3 b}{\partial \bar{z} \partial z^2} - \frac{\partial a}{\partial z} \frac{\partial^3 b}{\partial \bar{z}^2 \partial z} \right) + \left( \frac{\partial b}{\partial z} \frac{\partial^3 a}{\partial \bar{z}^2 \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^3 a}{\partial \bar{z} \partial z^2} \right) \right] \\
&\quad - 2(1 - z\bar{z})^3 \left[ z \left( \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial z^2} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial z^2} \right) + \left( \frac{\partial b}{\partial z} \frac{\partial^2 a}{\partial \bar{z} \partial z} - \frac{\partial a}{\partial z} \frac{\partial^2 b}{\partial \bar{z} \partial z} \right) \right) \right. \\
&\quad \left. + \bar{z} \left( \left( \frac{\partial^2 a}{\partial \bar{z}^2} \frac{\partial b}{\partial z} - \frac{\partial^2 b}{\partial \bar{z}^2} \frac{\partial a}{\partial z} \right) + \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial \bar{z} \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial \bar{z} \partial z} \right) \right) \right] \\
&\quad + 4z\bar{z}(1 - z\bar{z})^2 \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial a}{\partial z} \right) - 2(1 - z\bar{z})^2 \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial a}{\partial z} \right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{i}{8\pi^2} (\{a, \Delta b\} + \{\Delta a, b\}) &= (1 - z\bar{z})^4 \left[ \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^3 b}{\partial \bar{z} \partial z^2} - \frac{\partial a}{\partial z} \frac{\partial^3 b}{\partial \bar{z}^2 \partial z} \right) + \left( \frac{\partial b}{\partial z} \frac{\partial^3 a}{\partial \bar{z}^2 \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^3 a}{\partial \bar{z} \partial z^2} \right) \right] \\
&\quad - 2(1 - z\bar{z})^3 \left[ z \left( \frac{\partial b}{\partial z} \frac{\partial^2 a}{\partial \bar{z} \partial z} - \frac{\partial a}{\partial z} \frac{\partial^2 b}{\partial \bar{z} \partial z} \right) + \bar{z} \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial^2 b}{\partial \bar{z} \partial z} - \frac{\partial b}{\partial \bar{z}} \frac{\partial^2 a}{\partial \bar{z} \partial z} \right) \right].
\end{aligned}$$

Thus we have finally

$$\begin{aligned}
& (\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) \\
&= \frac{i\hbar}{2\pi} \{a, b\} + \frac{\hbar^2}{2} \left( \frac{i}{8\pi^2} \Delta \{a, b\} + \frac{i}{8\pi^2} \{a, \Delta b\} + \frac{i}{8\pi^2} \{\Delta a, b\} + \frac{i}{\pi} \{a, b\} \right) + o(\hbar^2) \\
&= i\hbar \{a, b\} + i \frac{\hbar^2}{4} (\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\} + 8\pi \{a, b\}) + o(\hbar^2). \tag{11.6}
\end{aligned}$$

The three-term asymptotic expansion of the commutator is obtained by substituting the expressions for partial derivatives of the function  $c_1$  calculated at the origin to (11.5) and by doing the symbolic calculations in MatLab 6.5. The final result is as follows:

$$\begin{aligned}
(\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) &= \frac{i\hbar}{2\pi} \{a, b\} \\
&+ \frac{\hbar^2}{2} \left[ \frac{i}{8\pi^2} (\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) + \frac{i}{\pi} \{a, b\} \right] \\
&+ \hbar^3 \left[ \frac{i}{192\pi^3} (\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\}) \right. \\
&+ \Delta^2 \{a, b\} + \Delta \{a, \Delta b\} + \Delta \{\Delta a, b\}) \\
&+ \left. \frac{7i}{48\pi^2} (\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) + \frac{i}{2\pi} \{a, b\} \right] + o(\hbar^3) \\
&= i\hbar \{a, b\} + i \frac{\hbar^2}{4} (\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\} + 8\pi \{a, b\}) \\
&+ i \frac{\hbar^3}{24} [\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2 \{a, b\} \\
&+ \Delta \{a, \Delta b\} + \Delta \{\Delta a, b\} + 28\pi (\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) \\
&+ 96\pi^2 \{a, b\}] + o(\hbar^3),
\end{aligned}$$

where the Poisson bracket  $\{, \}$  and the Laplace–Beltrami operator  $\Delta$  are given by (5.2) and by (5.3), respectively.

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