Integr. equ. oper. theory 50 (2004), 217–253
0378-620X/020217-37, DOI 10.1007/s00020-003-1295-z

Integral Equations and Operator Theory

Dynamics of Properties of Toeplitz Operators with Radial Symbols

S. Grudsky, A. Karapetyants and N. Vasilevski

Dedicated to the fond memory of Olga Grudskaia, who generously assisted in the preparation of the figures in this paper.

Abstract. In the case of radial symbols we study the behavior of different properties (boundedness, compactness, spectral properties, etc.) of Toeplitz operators $T_a^{(\lambda)}$ acting on weighted Bergman spaces $\mathcal{A}_\lambda^2(\mathbb{D})$ over the unit disk $\mathbb{D}$, in dependence on λ, and compare their limit behavior under $\lambda \to +\infty$ with corresponding properties of the initial symbol a.

Mathematics Subject Classification (2000). Primary 47B35; Secondary 47A25, 47B10, 46E22.

Keywords. Toeplitz operator, weighted Bergman spaces, boundedness, compactness, spectra.

1. Introduction

Toeplitz operators with smooth (or continuous) symbols acting on weighted Bergman spaces over the unit disk, as well as C^*-algebras generated by such operators, naturally appear in the context of problems in mathematical physics. We mention here only a few close to our area of interest: the quantum deformation of the algebra of continuous functions on the disk [11], and the Berezin quantization procedure (in particular, on the hyperbolic plane); see, for example, [2, 3, 4].

Given a smooth symbol $a = a(z)$, the family of Toeplitz operators $T_a = \{T_a^{(h)}\}$, with $h \in (0, 1)$, is considered under the Berezin quantization procedure [2, 3]. For a fixed h the Toeplitz operator $T_a^{(h)}$ acts on the weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$, where h is the parameter characterizing the weight on $\mathcal{A}_h^2(\mathbb{D})$. In the

This work was partially supported by CONACYT Project U40654-F, México.
The first author was partially supported by CONACYT Grant, Cátedra Patrimonial, Nivel II, No 010186.

special quantization procedure each Toeplitz operator $T_a^{(h)}$ is represented by its Wick symbol $\widetilde{a}_h$, and the correspondence principle says that for smooth symbols one has

$$\lim_{h\to 0} \widetilde{a}_h = a.$$

Moreover by [10] the above limit remains valid in the L_1-sense for a wider class of symbols.

The same, as in a quantization procedure, weighted Bergman spaces are appeared naturally in many questions of complex analysis and operator theory. In the last cases a weight parameter is normally denoted by λ and runs through $(-1, +\infty)$. In the sequel we will consider weighted Bergman spaces $\mathcal{A}^2_\lambda(\mathbb{D})$ parameterized by $\lambda \in (-1, +\infty)$ which is connected with $h \in (0, 1)$, used as the parameter in the quantization procedure, by the rule $\lambda + 2 = \frac{1}{h}$.

We study the behavior of different properties (boundedness, compactness, spectral properties, etc.) of $T_a^{(\lambda)}$ in dependence on λ, and compare of their limit behavior under $\lambda \to \infty$ with corresponding properties of the initial symbol a.

Although the word "dynamics" in the title may sound ambiguous, it is to emphasize the main theme of the paper: *what happens to properties of Toeplitz operators acting on weighted Bergman spaces when the weight parameter varies.*

In the article we consider the Toeplitz operators with only radial symbols $a = a(|z|)$. In this case each Toeplitz operator is unitary equivalent to a multiplication operator acting on the one-sided space l_2. This permits us to get more explicit information than can be obtained studying general Wick symbols.

By reasons of dimension it is quite obvious that smooth radial symbols form a commutative subalgebra of the Poisson algebra on the unit disk (= hyperbolic plane) equipped with the corresponding standard symplectic form. But it is important that their quantum counterparts, Toeplitz operators with radial symbols, form a *commutative* operator algebra as well. Moreover these commutative properties of Toeplitz operators do not depend at all on smoothness properties of symbols, the radial symbols can be merely measurable. The prime cause here is geometry. Radial symbols provide the most transparent example of functions which are *constant on cycles*, the orthogonal trajectories to the system of *geodesics intersecting in a single point.* The results for all other such cases can be easily obtained further by means of Möbius transformations.

Note in this connection (for details see [18, 19]) that all recently discovered cases of commutative *-algebras of Toeplitz operators on the unit disk are classified by pencils of geodesics of the following three possible types: geodesics intersecting in a single point (elliptic pencil), parallel geodesics (parabolic pencil), and disjoint geodesics, i.e., all geodesics orthogonal to a given one (hyperbolic pencil). Symbols which are constant on corresponding cycles (called cycles, horocycles, or hypercycles, depending on the pencil) generate in each case the commutative algebra of Toeplitz operators.

In this article we deal only with radial symbols, and thus only with the first case connected with elliptic pencils. The two remaining cases will be considered in forthcoming papers.

Note that Toeplitz operators with radial symbols have been intensively studied recently in a different context (see, for example, [7, 8, 12, 13, 14, 15, 16]).

We begin the article with the analysis of the boundedness and compactness properties. Although the existence of unbounded symbols for which the corresponding Toeplitz operators are bounded were known, it was rather a surprise that there exist *unbounded at each point of the boundary* symbols which generate *bounded* and even *compact* Toeplitz operators. This phenomenon, with examples for the case of radial symbols, was described in [8]. At this stage a number of natural questions emerge: *Describe the conditions on symbols for boundedness (compactness) of the corresponding Toeplitz operators. Given an unbounded symbol, describe the set of λ for which the corresponding Toeplitz operator is bounded (compact).* We devote Section 2 to these problems. In particular we describe all possible sets of λ for which a Toeplitz operator remains bounded (compact), and give examples showing that all these possibilities can be realized.

We devote Section 3 to a more delicate structure on the set of compact operators. We study here when Toeplitz operators with radial symbols belong to the Schatten classes.

In the next three sections we study spectral properties. Note that any Toeplitz operator with an radial symbol continuous at the point 1 and acting on each weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$ is just a compact perturbation of a multiple of identity; i.e., $T_a^{(\lambda)} = a(1)I + K_\lambda$, where K_λ is compact. The spectrum of each $T_a^{(\lambda)}$ consists of a discrete set of points, a sequence, tending to $a(1)$. For each fixed λ this sequence seems to be quite unrestricted, it has only to converge to $a(1)$. The definite tendency starts appearing as λ tends to infinity. The correspondence principle suggests that the limit set of those spectra has to be strictly connected with the range of the initial symbol. This is definitely true for continuous symbols. Given a symbol $a(r)$, the limit set of spectra, which we will denote by $M_\infty(a)$, does coincide with the range of $a(r)$. The new effects appear when we consider more complicated symbols. In particular, in the case of piecewise continuous symbols the limit set $M_\infty(a)$ coincides with the range of $a(r)$ together with the *line segments connecting the one-sided limit points* of our of piecewise continuous symbol.

Note that these additional line segments may essentially enlarge the limit set $M_\infty(a)$ comparing to the range of a symbol. For a measurable and, in general, unbounded symbol one always has

$$\operatorname{Range} a \subset M_\infty(a) \subset \operatorname{conv}(\operatorname{Range} a), \tag{1.1}$$

and the gap between these extreme sets can be substantial. We give a number of examples illustrating possible interrelations between sets in (1.1). In particular, we give two examples of radial symbols with $\operatorname{clos}(\operatorname{Range} a) = S^1 = \partial\mathbb{D}$ and with $\operatorname{Range} a = S^1$, such that in the first case $M_\infty(a) = \mathbb{D}$, while $M_\infty(a)$ is a countable union of circles whose radii tend to 1, in the second case. For unbounded symbols the limit set $M_\infty(a)$ may even coincide with the whole complex plane $\mathbb{C}$.

The examples considered in this paper have been chosen for mathematical, rather than physical interest. While the simplest possible examples would be provided by real symbols (generating self-adjoint Toeplitz operators), some simple complex-valued symbols have been chosen for the figures in order to illustrate spectra not confined to the real line, where the spectra tend to be indistinguishable.

2. Boundedness and compactness properties

Denote by $\mathbb{D}$ the unit disk in $\mathbb{C}$, and introduce the weighted Hilbert space $L_2(\mathbb{D}, d\mu_\lambda)$ which consists of measurable functions f on $\mathbb{D}$ for which the norm

$$\|f\|_{L_2(\mathbb{D},d\mu_\lambda)} = \left(\int_{\mathbb{D}} |f(z)|^2 d\mu_\lambda(z)\right)^{1/2}$$

finite. Here

$$d\mu_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda \frac{1}{\pi} dv(z), \qquad \lambda > -1,$$

where $dv(z) = dxdy$ is the Euclidian area element. The weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$ is the closed subspace of $L_2(\mathbb{D}, d\mu_\lambda)$ consisting of analytic functions. The orthogonal Bergman projection $B^{(\lambda)}_{\mathbb{D}}$ of $L_2(\mathbb{D}, d\mu_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{D})$ has the form

$$(B^{(\lambda)}_{\mathbb{D}} f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^{\lambda+2}}\, d\mu_\lambda(\zeta).$$

Given a function $a = a(z)$, the Toeplitz operator $T^{(\lambda)}_a$ with symbol a is defined on $\mathcal{A}^2_\lambda(\mathbb{D})$ as follows

$$T^{(\lambda)}_a : f \in \mathcal{A}^2_\lambda(\mathbb{D}) \longmapsto (B^{(\lambda)}_{\mathbb{D}} af)(z) \in \mathcal{A}^2_\lambda(\mathbb{D}).$$

From now on we will consider only radial symbols $a = a(|z|) = a(r)$. As was proved in [7], each Toeplitz operator $T^{(\lambda)}_a$ with radial symbols $a = a(r)$, acting on the weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$, is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, acting on the one-sided l_2. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ is given (see, [7], formula (3.1)) by

$$\gamma_{a,\lambda}(n) = \frac{1}{\mathrm{B}(n+1,\lambda+1)} \int_0^1 a(\sqrt{r})\,(1-r)^\lambda\, r^n dr, \qquad n \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}, \tag{2.1}$$

where $\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the usual Beta function.

Thus, in particular, the Toeplitz operator $T^{(\lambda)}_a$ with radial symbols $a = a(r)$ is bounded on $\mathcal{A}^2_\lambda(\mathbb{D})$ if and only if the sequence $\gamma_{a,\lambda}$ is bounded, and is compact on $\mathcal{A}^2_\lambda(\mathbb{D})$ if and only if the sequence $\gamma_{a,\lambda}$ has the zero limit.

Although the existence of unbounded symbols for which the corresponding Toeplitz operators are bounded were known, it was rather a surprise that there exist *unbounded at each point of the boundary* symbols which generate *bounded* and even *compact* Toeplitz operators. The analysis of this phenomenon for the case of radial symbols was given in [8], where, in particular, the existence of *unbounded at the boundary* symbols which generate *compact* Toeplitz operators was shown.

For every bounded symbol $a(r)$ the Toeplitz operator $T_a^{(\lambda)}$ is bounded on all spaces $\mathcal{A}^2_\lambda(\mathbb{D})$, where $\lambda \in (-1,\infty)$, and the corresponding norms are uniformly bounded by $\sup_r |a(r)|$. As examples show, a Toeplitz operator with unbounded symbol can be bounded (compact) for one value of λ and unbounded (non compact) for another. At this stage the natural question appears: *given an unbounded symbol, describe the set of λ for which the corresponding Toeplitz operator is bounded (compact).*

As has been already mentioned, all spaces $\mathcal{A}^2_\lambda(\mathbb{D})$, where $\lambda \in (-1,\infty)$, are natural and appropriate for Toeplitz operators with *bounded* symbols. Admitting *unbounded* symbols and wishing to have a sufficiently large class of them common for all admissible λ, it is convenient to study corresponding Toeplitz operators on $\mathcal{A}^2_\lambda(\mathbb{D})$ with $\lambda \in [0,\infty)$.

In the sequel we consider radial symbols $a(r)$ satisfying the condition $a(\sqrt{r}) \in L_1(0,1)$, and assume that $\lambda \in [0,\infty)$.

We start with sufficient conditions for boundedness of Toeplitz operators with unbounded radial symbols.

Given symbol $a(\sqrt{r}) \in L_1(0,1)$, introduce $B_a^{(0)}(r) = a(\sqrt{r})$. Then (for $n \geq 1$)

$$\begin{aligned}
\gamma_{a,\lambda}(n) &= \frac{1}{B(n+1,\lambda+1)} \int_0^1 a(\sqrt{r})(1-r)^\lambda r^n dr \\
&= \frac{1}{B(n+1,\lambda+1)} \int_0^1 a(\sqrt{r})dr \int_0^r \omega_\lambda(n,s)ds \\
&= \frac{1}{B(n+1,\lambda+1)} \int_0^1 \omega_\lambda(n,s)ds \int_s^1 a(\sqrt{r})dr \\
&= \frac{1}{B(n+1,\lambda+1)} \int_0^1 B_a^{(1)}(s)\omega_\lambda(n,s)ds.
\end{aligned}$$

We use Fubini's theorem when changing the order of integration. Here we have denoted

$$B_a^{(1)}(s) = \int_s^1 a(\sqrt{r})dr \quad \text{and} \quad \omega_\lambda(n,s) = \frac{d}{ds}((1-s)^\lambda s^n).$$

Studying the boundedness of Toeplitz operators with radial symbols we need to analyze the boundedness of corresponding sequences $\gamma_{a,\lambda}$. Under the assumptions made each finite set of their initial elements is bounded, we need to study only the limit behaviour or their elements. Thus without a loss of generality we may always assume that n is large enough.

Integrating by parts and neglecting the boundary terms, we get (for $n \geq j$)

$$\gamma_{a,\lambda}(n) = \frac{1}{B(n+1,\lambda+1)} \int_0^1 B_a^{(j)}(s)\ \frac{d^{j-1}}{ds^{j-1}}\ \omega_\lambda(n,s)\ ds, \quad j = 0, 1, \ldots, \tag{2.2}$$

where

$$B_a^{(j)}(s) = \int_s^1 B^{(j-1)}(r)\, dr, \qquad j = 1, 2, \ldots . \tag{2.3}$$

Theorem 2.1. *If there exists $j \in \mathbb{N}$ such that*

$$B_a^{(j)}(r) = O((1-r)^j), \qquad r \to 1, \tag{2.4}$$

then the Toeplitz operator $T_a^{(\lambda)}$ is bounded on each $\mathcal{A}_\lambda^2(\mathbb{D})$, with $\lambda \geq 0$.

If for some $j \in \mathbb{N}$

$$B_a^{(j)}(r) = o((1-r)^j), \qquad r \to 1, \tag{2.5}$$

then the Toeplitz operator $T_a^{(\lambda)}$ is compact on each $\mathcal{A}_\lambda^2(\mathbb{D})$, with $\lambda \geq 0$.

Proof. To prove the boundedness we show that the sequence (2.1) is bounded.

Consider first the case $j = 1$. Integrating by parts and using (2.4) we have

$$\begin{aligned} |\gamma_{a,\lambda}(n)| &= \left| \frac{1}{\mathrm{B}(n+1,\lambda+1)} \int_0^1 B_a^{(1)}(r)\, \omega_\lambda(n,r) dr \right| \\ &\leq \text{const}\, \frac{1}{\mathrm{B}(n+1,\lambda+1)} \left(\lambda \int_0^1 (1-r)^\lambda\, r^n dr + n \int_0^1 (1-r)^{\lambda+1}\, r^{n-1} dr \right) \\ &= \text{const} \left(\lambda \frac{\mathrm{B}(n+1,\lambda+1)}{\mathrm{B}(n+1,\lambda+1)} + n \frac{\mathrm{B}(n,\lambda+2)}{\mathrm{B}(n+1,\lambda+1)} \right) \\ &= \text{const}\, (\lambda + (\lambda+1)) = \text{const}\, (2\lambda+1). \end{aligned}$$

The case $j > 1$ are considered analogously applying consecutive integration by parts.

Sufficiency of the condition (2.5) for compactness requires more delicate but similar arguments. □

We illustrate the theorem giving an example, all details of which can be found in [8].

Example 2.2. Consider unbounded symbol

$$a(r) = (1-r^2)^{-\beta} \sin(1-r^2)^{-\alpha}, \tag{2.6}$$

where $\alpha > 0$, and $0 < \beta < 1$. By Theorem 2.1 with $j = 1$ the corresponding Toeplitz operator

- is bounded for $\alpha \geq \beta$;
- is compact for $\alpha > \beta$.

The conditions of Theorem 2.1 are only sufficient for oscillating symbols, and the above conclusions are in fact just preliminary. The complete information about the symbol (2.6) is given in Example 3.6 below and says that the corresponding Toeplitz operator is bounded and compact for all $\lambda \geq 0$, $\alpha > 0$, and $0 < \beta < 1$.

The next theorem deals with necessary conditions for boundedness and compactness.

Theorem 2.3. *Let $a(\sqrt{r}) \in L_1(0,1)$, and either $a(r) \geq 0$, or $B_a^{(j)}(r) \geq 0$ for a certain $j \in \mathbb{N}$. Then the conditions (2.4), (2.5) are also necessary for the boundedness and compactness of the corresponding Toeplitz operator $T_a^{(\lambda)}$ on $\mathcal{A}_\lambda^2(\mathbb{D})$ with $\lambda \geq 0$, respectively. That is, the conditions (2.4), (2.5) are necessary and sufficient for boundedness and respectively compactness of the Toeplitz operator $T_a^{(\lambda)}$ with the symbol $a = a(r)$ as specified on $\mathcal{A}_\lambda^2(\mathbb{D})$, where $\lambda \geq 0$.*

Proof. We prove that these conditions are necessary when $a(r) \geq 0$. The general case of $B_a^{(j)}(r) \geq 0$ for a certain $j \in \mathbb{N}$ can be covered similarly.

For $\lambda = 0$ and arbitrary $s \in (0,1)$ we set $n = \left[\frac{1}{1-s}\right] - 1$. Taking into account that $B(n+1, 1 = \frac{1}{n+1}$, we have

$$\gamma_{a,0}(n) \geq (n+1) \int_s^1 a(\sqrt{r})dr \geq \frac{1}{2}\,(1-s)^{-1} B_a^{(1)}(s),$$

which proves the statement for $\lambda = 0$. Let now $\lambda = \lambda_0 > 0$. Below we show (Theorem 2.7) that if γ_{a,λ_0} is bounded (compact) for some λ_0, then it is also bounded (compact) for each $\lambda \in [0, \lambda_0)$. □

Corollary 2.4. *If $a(r) \geq 0$, and $\lim_{\varepsilon\to 0} \inf_{r\in[1-\varepsilon,1]} a(r) = +\infty$, then the Toeplitz operator $T_a^{(\lambda)}$ is unbounded on each $\mathcal{A}_\lambda^2(\mathbb{D})$, $\lambda \geq 0$.*

Corollary 2.5. *Let $a(\sqrt{r}) \in L_1(0,1)$, and let $a(r) \geq 0$, or $B_a^{(j)}(r) \geq 0$ for some $j \in \mathbb{N}$. Then the Toeplitz operator $T_a^{(\lambda)}$ is bounded (compact), or unbounded (not compact) on each $\mathcal{A}_\lambda^2(\mathbb{D})$ simultaneously.*

To prove the corollary it is sufficient to note that the above conditions for boundedness and compactness do not depend on λ.

We would like to mention in this connection a result by Kehe Zhu ([20]) who proved the assertion of Corollary 2.5 for nonnegative symbols for Toeplitz operators acting on the weighted Bergman space over the unit ball in $\mathbb{C}^n$). Our statement is not limited to only nonnegative symbols.

Remark 2.6. For general $a(\sqrt{r}) \in L_1(0,1)$ symbols the conditions (2.4), (2.5) fail to be necessary. Further, while j becomes large these conditions get weaker. But the hypothesis that each operator $T_a^{(\lambda)}$ $(a = a(r))$ bounded (or compact) on $\mathcal{A}_\lambda^2(\mathbb{D})$ will satisfy (2.4) (or (2.5)) for some large j appears to be false.

Indeed, there exists (see Theorem 2.8 below) a Toeplitz operator $T_a^{(\lambda)}$, bounded (compact) on $\mathcal{A}^2_{\lambda_1}(\mathbb{D})$ and unbounded (not compact) on $\mathcal{A}^2_{\lambda_2}(\mathbb{D})$, for some $\lambda_1 < \lambda_2$, while the conditions (2.4) and (2.5) do not depend on λ. Furthermore, Example 3.6 below illustrates dependence of the conditions (2.4) and (2.5) on j.

We pass now to the analysis of the boundedness and compactness properties in their dependence on $\lambda \in [0, \infty)$.

Theorem 2.7. *The following statements hold:*

(i) *if for any $\lambda_0 > 0$, the sequence γ_{a,λ_0} is bounded, then the sequence $\gamma_{a,\lambda}$ is bounded for all $\lambda \in [0, \lambda_0)$;*

(ii) *if for any $\lambda_0 > 0$, $\lim_{n\to\infty} \gamma_{a,\lambda_0}(n) = 0$, then $\lim_{n\to\infty} \gamma_{a,\lambda_0}(n) = 0$ for all $\lambda \in [0, \lambda_0)$.*

Proof. Introduce

$$\beta_{a,\lambda}(n) = \int_0^1 a(\sqrt{r})\,(1-r)^\lambda\, r^n dr.$$

and for $\sigma < 0$ introduce the power series

$$(1-r)^\sigma = \sum_{j=0}^{\infty} C_j(\sigma)\, r^j,$$

The coefficients $C_j(\sigma)$ are given (see, for example, [1] p. 256) in the form

$$\begin{aligned} C_j(\sigma) &= (-1)^j \frac{\Gamma(1+\sigma)}{\Gamma(j+1)\Gamma(1+\sigma-j)} = \frac{(-1)^j}{\pi}\,\frac{\Gamma(1+\sigma)\Gamma(j-\sigma)}{\Gamma(j+1)}\sin\pi(j-\sigma) \\ &= -\frac{1}{\pi}\,\sin\pi\sigma\,\frac{\Gamma(1+\sigma)\Gamma(j-\sigma)}{\Gamma(j+1)} = -\frac{1}{\pi}\,\sin\pi\sigma\,\mathrm{B}(j-\sigma, 1+\sigma). \end{aligned} \tag{2.7}$$

We note that for $-\sigma \in \mathbb{N}$ the above formula is understood in the limit sense, i.e.,

$$C_j(\sigma) = -\frac{1}{\pi}\lim_{s\to\sigma} \sin\pi s\,\mathrm{B}(j-s, 1+s).$$

Thus for $\lambda < \lambda_0$ we have

$$\begin{aligned} \beta_{a,\lambda}(n) &= \int_0^1 a(\sqrt{r})\,(1-r)^{\lambda_0}\,(1-r)^{\lambda-\lambda_0}\, r^n dr \\ &= -\sum_{j=0}^{\infty} \frac{\sin\pi(\lambda-\lambda_0)}{\pi}\,\mathrm{B}(j-\sigma, 1+\sigma)\int_0^1 a(\sqrt{r})\,(1-r)^{\lambda_0}\, r^{n+j} dr \\ &= \frac{\sin\pi(\lambda_0-\lambda)}{\pi}\sum_{j=0}^{\infty} \mathrm{B}(j-\sigma, 1+\sigma)\,\beta_{a,\lambda_0}(n+j), \end{aligned}$$

where $\sigma = \lambda - \lambda_0 < 0$.

Now it is easy to see that

$$\gamma_{a,\lambda}(n) = \frac{\sin\pi|\sigma|}{\pi}\sum_{j=0}^{\infty} \frac{\mathrm{B}(j-\sigma, 1+\sigma)\,\mathrm{B}(n+j+1, \lambda_0+1)}{\mathrm{B}(n+1, \lambda+1)}\,\gamma_{a,\lambda_0}(n+j). \tag{2.8}$$

We need the following asymptotic representation of the Beta function for fixed δ and $L \to \infty$

$$\begin{aligned} \mathrm{B}(L,\delta) &= \frac{\Gamma(L)\,\Gamma(\delta)}{\Gamma(L+\delta)} = \Gamma(\delta)\,\frac{e^{-L}L^{L-\frac{1}{2}}(1+O(L^{-1}))}{e^{-(L+\delta)}(L+\delta)^{L+\delta-\frac{1}{2}}(1+O(L^{-1}))} \\ &= \Gamma(\delta)\,e^{\delta}\left(1-\frac{\delta}{L+\delta}\right)^{L-\frac{1}{2}}(L+\delta)^{-\delta}\,(1+O(L^{-1})) \\ &= \Gamma(\delta)\,L^{-\delta}\,(1+O(L^{-1})). \end{aligned} \tag{2.9}$$

Let now the sequence γ_{a,λ_0} be bounded. Then

$$\begin{aligned} |\gamma_{a,\lambda}(n)| &\le \text{const} \sum_{j=0}^{\infty} \frac{(n+1)^{\lambda+1}}{(j-\sigma)^{1+\sigma}\,(n+j+1)^{\lambda_0+1}} \\ &\le \text{const}\,(n+1)^{\lambda+1}\int_0^{\infty}\frac{ds}{(s+1)^{1+\sigma}\,(n+s+1)^{\lambda_0+1}} \\ &\le \text{const}\,\frac{(n+1)^{\lambda+1}}{n^{\lambda+1}}\int_0^{\infty}\frac{dv}{v^{1+(\lambda-\lambda_0)}\,(1+v)^{\lambda_0+1}}, \end{aligned}$$

where $s+1=nv$.

Since $1+(\lambda-\lambda_0)<1$ and $(1+(\lambda-\lambda_0))+(\lambda_0+1)>1$, the last integral converges, and we have (i). To prove (ii) it is sufficient to observe that if $\lim_{n\to\infty}\gamma_{a,\lambda_0}(n)=0$, then const $=c(n)$ tends to 0 when $n\to\infty$ as well. $\Box$

Given a symbol $a=a(r)$, denote by $B(a)$ the set of all $\lambda\in[0,\infty)$ for which the Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}^2_\lambda(\mathbb{D})$, that is, the sequence $\gamma_{a,\lambda}$ is bounded; and denote by $K(a)$ the set of all $\lambda\in[0,\infty)$ for which the Toeplitz operator $T_a^{(\lambda)}$ is compact on $\mathcal{A}^2_\lambda(\mathbb{D})$, that is the sequence $\gamma_{a,\lambda}$ has the zero limit.

Theorem 2.7 shows that the sets $B(a)$ and $K(a)$ may have one of the following forms:

$$(i)\ \ [0,\infty), \qquad (ii)\ \ [0,\lambda_0), \qquad (iii)\ \ [0,\lambda_0].$$

We show now that all three possibilities can be realized. The first one is true, for example, for any bounded symbol, in the case of $B(a)$, and for any continuous bounded symbol with $a(1)=0$, in the case of $K(a)$. To realize (ii) and (iii) consider the sequence

$$\gamma(n)=e^{\frac{i}{5\pi}\ln^2(n+1)}\,\ln^{-\nu}(n+1)\,\ln^{\beta}\ln(n+1),$$

where $\nu>0$ and $\beta\in\mathbb{R}$. Analogously to Example 5.10 from [7] it can be shown that there exists a radial function (symbol) $a_{\nu,\beta}(r)$ such that

$$\gamma_{a_{\nu,\beta},0}(n)=\gamma(n), \qquad n\in\mathbb{Z}_+.$$

The function $a_{\nu,\beta}(r)$ is recovered from

$$\begin{aligned}
\gamma_{a_{\nu,\beta},0}(n) &= (n+1)\int_0^1 a_{\nu,\beta}(\sqrt{r})r^n dr \\
&= (n+1)\int_0^\infty a_{\nu,\beta}(\sqrt{e^{-y}})e^{-(n+1)y}dy \\
&= i(n+1)\,\frac{1}{\sqrt{2\pi}}\int_0^\infty \left[-i\sqrt{2\pi}\,a_{\nu,\beta}(\sqrt{e^{-y}})e^{-y}\right] e^{i(in)y}dy,
\end{aligned}$$

considered as the inverse Fourier transform of the function $-i\sqrt{2\pi}\,a_{\nu,\beta}(\sqrt{e^{-y}})e^{-y} \in L_2(0,\infty)$, multiplied by $(x+i)$ and then calculated at the point in.

The following theorem gives an example of different behavior of $B(a)$ and $K(a)$.

Theorem 2.8. *Let* $0<\nu<1$. *Then*

a)	$B(a_{\nu,0}) = [0,\nu]$,	$K(a_{\nu,0}) = [0,\nu)$,	$\beta = 0$,
b)	$B(a_{\nu,\beta}) = [0,\nu)$,	$K(a_{\nu,\beta}) = [0,\nu)$,	$\beta > 0$,
c)	$B(a_{\nu,\beta}) = [0,\nu]$,	$K(a_{\nu,\beta}) = [0,\nu]$,	$\beta < 0$.

Proof. Let $0<\lambda<1$. Then

$$\begin{aligned}
\gamma_{a_{\nu,\beta},\lambda}(n) &= \frac{1}{B(n+1,\lambda+1)}\int_0^1 a_{\nu,\beta}(\sqrt{r})((1-r)(1-r)^{\lambda-1}\,r^n dr \\
&= \frac{1}{B(n+1,\lambda+1)}\sum_{j=0}^\infty C_j(\lambda-1)\int_0^1 a_{\nu,\beta}(\sqrt{r})(1-r)r^{n+j}dr \\
&= \frac{1}{B(n+1,\lambda+1)}\sum_{j=0}^\infty C_j(\lambda-1)(\beta_{a_{\nu,\beta},0}(n+j)-\beta_{a_{\nu,\beta},0}(n+j+1))
\end{aligned}$$

where

$$\beta_{a_{\nu,\beta},0}(n) = \frac{\gamma_{a_{\nu,\beta},0}(n)}{n+1}. \tag{2.10}$$

Further

$$\gamma_{a_{\nu,\beta},\lambda}(n) = \frac{1}{B(n+1,\lambda+1)}\sum_{j=0}^\infty C_j(\lambda-1)\int_j^{j+1}\beta'_{a_{\nu,\beta},0}(n+s)\,ds. \tag{2.11}$$

It is easy to see that

$$\begin{aligned}
\beta'_{a_{\nu,\beta},0}(s) &= \frac{2i}{5\pi}\cdot\frac{e^{\frac{i}{5\pi}\ln^2(n+s+1)}}{(n+s+1)^2}\ln^{1-\nu}(n+s+1)\ln^\beta\ln(n+s+1) \\
&+ O\left(\frac{\ln^{-\nu}(n+s+1)}{(n+s+1)^2}\ln^\beta\ln(n+s+1)\right) \\
&:= \delta_0(n+s)+\delta_1(n+s).
\end{aligned} \tag{2.12}$$

By (2.7)

$$C_j(\lambda-1) = \frac{1}{\pi}\sin(\pi(1-\lambda))B(j+1-\lambda,\lambda),$$

and thus (2.9) implies that

$$C_j(\lambda-1) = \sin\pi(1-\lambda)\,\frac{\Gamma(\lambda)}{\pi}\,(j+1-\lambda)^{-\lambda}\,(1+O((j+1)^{-1}). \qquad (2.13)$$

Thus from (2.11)–(2.13) we get

$$\begin{aligned}
\gamma_{a_{\nu,\beta},\lambda}(n) &= \frac{c_\lambda}{B(n+1,\lambda+1)}\sum_{j=0}^{\infty}\int_j^{j+1}\frac{\delta_0(n+s)}{(s+1)^\lambda}\,ds \\
&+ \frac{1}{B(n+1,\lambda+1)}\sum_{j=0}^{\infty}O\left(\frac{\delta_1(n+j+1)}{(j+1)^\lambda}\right) \\
&+ \frac{1}{B(n+1,\lambda+1)}\sum_{j=0}^{\infty}O\left(\frac{\delta_0(n+j+1)}{(j+1)^{1+\lambda}}\right) \\
&= \frac{c_\lambda}{B(n+1,\lambda+1)}\int_0^{\infty}\frac{\delta_0(s+n)}{(s+1)^\lambda}\,ds \\
&+ (n+1)^{\lambda+1}\sum_{j=0}^{\infty}O\left(\frac{\delta_1(n+j+1)}{(j+1)^\lambda}\right) \\
&+ (n+1)^{\lambda+1}\sum_{j=0}^{\infty}O\left(\frac{\delta_0(n+s+1)}{(j+1)^{1+\lambda}}\right) := I_1+I_2+I_3
\end{aligned}$$

where $c_\lambda = \frac{\sin\pi(1-\lambda)}{\pi}\Gamma(\lambda)$.

We estimate now summands I_2 and I_3.

$$I_2 \le \text{const}\,(n+1)^{1+\lambda}\int_0^{\infty}\frac{\ln^{-\nu}(n+s+1)\ln^{\beta}\ln(n+s+1)}{(n+s+1)^2\,(s+1)^\lambda}\,ds.$$

Changing the variable $s+1=nv$ we have

$$\begin{aligned}
I_2 &\le \text{const}\int_0^{\infty}\frac{\ln^{-\nu}n(v+1)\ln^{\beta}\ln n(v+1)}{(1+v)^2\,v^\lambda}\,dv \\
&\le \text{const}\,\ln^{-\nu}n\,\ln^{\beta}\ln n\int_0^{\infty}\frac{\left(1+\frac{\ln(1+v)}{\ln n}\right)^{-\nu}\left(1+\frac{\ln\left(1+\frac{\ln(1+v)}{\ln n}\right)}{\ln\ln n}\right)^{\beta}}{(1+v)^2\,v^\lambda}\,dv \\
&\le \text{const}\,\ln^{-\nu}n\,\ln^{\beta}\ln n\int_0^{\infty}\frac{(1+\ln(1+v))^{|\beta|}dv}{(1+v)^2\,v^\lambda} \\
&\le \text{const}\,\ln^{-\nu}n\,\ln^{\beta}\ln n, \qquad (2.14)
\end{aligned}$$

noting that since $0\le\lambda<1$ then $2+\lambda>1$.

Now we pass to the evaluation of I_3:

$$I_3 \le \text{const}\,(n+1)^{\lambda+1}\int_0^{\infty}\frac{\ln^{1-\nu}(n+s+1)\ln^{\beta}\ln(n+s+1)}{(n+s+1)^2\,(s+1)^{1+\lambda}}\,ds.$$

Again changing the variable $s+1=nv$ we have

$$\begin{aligned} I_3 &\leq \text{const}\,\frac{1}{n}\int_{n^{-1}}^{\infty}\frac{\ln^{1-\nu} n(1+v)\ \ln^{\beta}\ln n(1+v)}{(1+v)^2v^{1+\lambda}}\,dv \\ &\leq \text{const}\,\frac{\ln^{1-\nu} n\ \ln^{\beta}(\ln n)}{n}\int_{n^{-1}}^{\infty}\frac{dv}{v^{1+\lambda}} \\ &\leq \text{const}\,\frac{\ln^{1-\nu} n\ \ln^{\beta}\ln n}{n^{1-\lambda}}. \end{aligned} \tag{2.15}$$

Thus we have (see (2.14) and (2.15))

$$I_2 = O(\ln^{-\nu} n\ \ln^{\beta}\ln n) \tag{2.16}$$

and

$$I_3 = O\left(\frac{\ln^{1-\nu} n\ \ln^{\beta}\ln n}{n^{1-\lambda}}\right). \tag{2.17}$$

Now we consider the "main" term

$$\begin{aligned} I_1 &= \frac{\widetilde{c}_\lambda}{B(n+1,\lambda+1)}\int_0^{\infty}\frac{e^{\frac{i}{5\pi}\ln^2(n+s+1)}\ln^{1-\nu}(n+s+1)\ln^{\beta}\ln(m+s+1)}{(m+s+1)^2(s+1)^{\lambda}}\,ds \\ &= \frac{\widetilde{c}_\lambda}{B(n+1,\lambda+1)n^{1+\lambda}}\int_0^{\infty}\frac{e^{\frac{i}{5\pi}\ln^2 n(v+1)}\ln^{1-\nu} n(1+v)\ln^{\beta}\ln(n(1+v))}{(1+v)^2v^{\lambda}}\,dv, \end{aligned}$$

where $\widetilde{c}_\lambda = \frac{2i}{5\pi}c_\lambda$.

Using the asymptotics of the B-function we have

$$\begin{aligned} I_1 &= \widetilde{c}_\lambda\Gamma(1+\lambda)(1+O(n^{-1}))e^{\frac{i}{5\pi}\ln^2 n}\ln^{1-\nu} n\ \ln^{\beta}(\ln n) \\ &\times \int_0^{\infty}\frac{e^{i\frac{2\pi}{5}(\ln n)\ln(1+v)}e^{\frac{i}{5\pi}\ln^2(v+1)}}{(v+1)^2\,v^{\lambda}} \\ &\quad \frac{\left(1+\frac{\ln(1+v)}{\ln n}\right)^{1-\nu}\left(1+\frac{\ln\left(1+\frac{\ln(1+v)}{\ln n}\right)}{\ln\ln n}\right)^{\beta}}{(v+1)^2\,v^{\lambda}}\,dv. \end{aligned} \tag{2.18}$$

Introduce a large parameter $\Lambda = \frac{2\pi}{5}\ln n$. We have the oscillatory integral with two points of singularity: $v_0=0$ and $v_0=\infty$. Let

$$1 = \chi_1(v)+\chi_2(v) \tag{2.19}$$

with $\chi_{1,2}(v)\in C^{\infty}(0,\infty)$, $\operatorname{supp}\chi_1\subset(0,1)$ and $\operatorname{supp}\chi_2\subset(1/2,\infty)$. According to this representation denote

$$I_1 := I_{1,1}+I_{1,2}.$$

Now $I_{1,1}$ can be written in the form

$$\begin{aligned} I_{1,1} &= \widetilde{c}_\lambda\Gamma(1+\lambda)(1+O(n^{-1}))e^{\frac{i}{5\pi}ln^2 n}\ln^{1-\nu} n\ln^{\beta}(\ln n) \\ &\times \left(\int_0^1\frac{e^{i\Lambda\ln(1+v)}F(v)}{v^{\lambda}}\,dv+\int_0^1 K(v,n)\,dv\right), \end{aligned} \tag{2.20}$$

where $F(v) = \frac{e^{\frac{i}{5\pi}\ln^2(v+1)}}{(1+v)^2}\chi_1(v)$, and the function $K(v,n)$ admits the following estimate

$$|K(v,n)| \leq \text{const}\,\frac{v}{(1+v)^2 v^\lambda \ln n(\ln\ln n)} = \frac{\text{const}\, v^{1-\lambda}}{\ln n(\ln\ln n)}.$$

Thus

$$\int_0^1 |K(v,n)|\,dv \leq \frac{\text{const}}{\ln n(\ln\ln n)}. \tag{2.21}$$

According to the stationary phase method (see, for example, [5], p.97) we have

$$\begin{aligned}\int_0^1 \frac{e^{i\Lambda\ln(1+v)}}{v^\lambda}F(v)\,dv &= \frac{F(0)\Gamma(1-\lambda)e^{i\frac{\pi}{2}(1-\lambda)}}{\Lambda^{1-\lambda}}\left(1+\frac{1}{\Lambda}\right)\\ &= \frac{d_\lambda}{\ln^{1-\lambda} n}\left(1+O\left(\frac{1}{\ln n}\right)\right)\end{aligned}$$

where

$$d_\lambda = \frac{\Gamma(1-\lambda)e^{i\frac{\pi}{2}(1-\lambda)}}{(2\pi/5)^{1-\lambda}}. \tag{2.22}$$

Thus from (2.21) and (2.22) we have

$$I_{1,1} = \widetilde{c}_\lambda d_\lambda \Gamma(1+\lambda)e^{\frac{i}{5\pi}\ln^2 n}\,\ln^{\lambda-\nu} n\ln^\beta(\ln n)\left(1+O\left(\frac{1}{\ln^\lambda n(\ln\ln\lambda)}\right)\right). \tag{2.23}$$

Pass now to the term $I_{1,2}$. Represent it in the form

$$I_{1,2} = \widetilde{c}_\lambda\Gamma(1+\lambda)(1+O(n^{-1}))e^{\frac{i}{5\pi}\ln^2 n}\ln^{1-\nu} n\,\ln^\beta(\ln n)\int_{1/2}^\infty e^{i\Lambda\ln(1+v)}\Phi(v,n)\,dv.$$

Integrating by parts we have

$$\begin{aligned}I_{1,2} &= \widetilde{c}_\lambda\Gamma(1+\lambda)(1+O(n^{-1}))e^{\frac{i}{5\pi}\ln^2 n}\ln^{1-\nu} n\,\ln^\beta(\ln n)\\ &\times\ \frac{1}{i\Lambda}\int_{1/2}^\infty e^{i\Lambda\ln(1+v)}\left[(1+v)\frac{\partial\Phi}{\partial v}(v,n)+\Phi(v,n)\right]dv.\end{aligned}$$

Since

$$|\Phi| + \left|(1+v)\frac{\partial\Phi}{\partial v}\right| \leq M\frac{\ln^{1-\nu} v\,\ln^{|\beta|}(\ln v)}{v^{2+\lambda}},$$

where M is independent of n, we have

$$I_{1,2} = O(\ln^{-\nu} n\,\ln^\beta(\ln n)). \tag{2.24}$$

Thus from (2.23) and (2.24) we get

$$I_1 = \widetilde{c}_\lambda d_\lambda\Gamma(1+\lambda)e^{\frac{i}{5\pi}\ln^2 n}\ln^{\lambda-\nu} n\,\ln^\beta(\ln n)\left(1+O\left(\frac{1}{\ln^\lambda n}\right)\right). \tag{2.25}$$

Thus finally (2.16), (2.17) and (2.25) show that

$$\gamma_{a_{\nu,\beta,\lambda}}(n) = \widetilde{c}_\lambda d_\lambda\Gamma(1+\lambda)e^{i\ln^2 n}\ln^{\lambda-\nu} n\,\ln^\beta(\ln n)\left(1+O\left(\frac{1}{\ln^\lambda n}\right)\right),$$

which implies all the statements of the theorem. □

3. Schatten classes

In many questions of operator theory, separation in the ideal of all compact operators of certain specific classes plays an important role. A special role is played by the so-called Schatten classes, and among them, the trace and Hilbert-Schmidt classes.

It is evident that Toeplitz operator $T_a^{(\lambda)}$ with radial symbol $a(r)$ belongs to Schatten class $K_p(\lambda)$, $1 \leq p < \infty$, on the space $\mathcal{A}^2_\lambda(\mathbb{D})$ if and only if

$$\|T_a^{(\lambda)}\|_{p,\lambda} = \left(\sum_{n=1}^{\infty} |\gamma_{a,\lambda}(n)|^p\right)^{1/p} < \infty.$$

In the context of this paper the following question is very natural: *given a radial symbol $a(r)$, what is the structure of the pairs (p,λ) for which $T_a^{(\lambda)} \in K_p(\lambda)$?*

We start with the sufficient conditions on a symbol $a(r)$ for $T_a^{(\lambda)} \in K_p(\lambda)$.

Theorem 3.1. *Let $a(\sqrt{r}) \in L_1(0,1)$ and let for some $j = 0, 1, \ldots,$ the function $B_a^{(j)}(r)$ (see (2.3)) satisfy one of the following conditions*

$$\int_0^1 |B_a^{(j)}(r)|(1-r)^{-(1+j+\frac{1}{p})}\, dr \quad < \quad \infty, \qquad p \geq 1, \tag{3.1}$$

$$\int_0^1 |B_a^{(j)}(r)|^p(1-r)^{-(2+j-\varepsilon)}\, dr \quad < \quad \infty, \qquad p > 1, \tag{3.2}$$

where $\varepsilon > 0$ can be arbitrarily small. Then $T_a^{(\lambda)} \in K_p(\lambda)$.

Proof. Assume first that $j = 0$, $p > 1$, and the condition (3.1) holds. Then

$$\|T_a^{(\lambda)}\|_{p,\lambda} = \left(\sum_{n=0}^{\infty} \left|\frac{1}{B(n+1,\lambda+1)}\int_0^1 a(\sqrt{r})(1-r)^\lambda r^n dr\right|^p\right)^{\frac{1}{p}}.$$

Applying the Hölder inequality we have

$$\begin{aligned}
\|T_a^{(\lambda)}\|_{p,\lambda} &= \left(\sum_{n=0}^{\infty}\left|\frac{1}{B(n+1,\lambda+1)}\int_0^1 \left(a^{\frac{1}{p}}(\sqrt{r})(1-r)^{(\lambda+1-\frac{1}{p^2})}r^n\right)\right.\right.\\
&\times \left.\left.\left(a^{\frac{1}{q}}(\sqrt{r})(1-r)^{-(1-\frac{1}{p^2})}dr\right)\right|^p\right)^{\frac{1}{p}}\\
&\le \left(\sum_{n=0}^{\infty}\frac{1}{B^p(n+1,\lambda+1)}\int_0^1 |a(\sqrt{r})|(1-r)^{(\lambda+1-\frac{1}{p^2})p}r^{np}dr\right.\\
&\times \left.\left(\int_0^1 |a(\sqrt{r})|(1-r)^{-q(1-\frac{1}{p^2})}dr\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\\
&\le \left(\int_0^1 |a(\sqrt{r})|(1-r)^{-(1+\frac{1}{p})}dr\right)^{\frac{1}{q}}\\
&\times \left(\int_0^1 |a(\sqrt{r})|(1-r)^{(\lambda+1-\frac{1}{p^2})p}\sum_{n=0}^{\infty}\frac{r^{np}}{B^p(n+1,\lambda+1)}\,dr\right)^{\frac{1}{p}} \qquad (3.3)
\end{aligned}$$

Estimate now the last sum using the asymptotic representation (2.9),

$$\begin{aligned}
\left|\sum_{n=0}^{\infty}\frac{r^{np}}{B^p(n+1,\lambda+1)}\right| &\le \text{const}\sum_{n=0}^{\infty} n^{p(\lambda+1)}\,r^{np}\\
&\le \text{const}\int_0^{\infty} u^{p(\lambda+1)}\,r^{up}\,du\\
&= \text{const}\int_0^{\infty} e^{-up\ln r^{-1}}\,u^{p(\lambda+1)}\,du\\
&= \text{const}\,(p\ln r^{-1})^{-p(\lambda+1)-1}\int_0^{\infty} e^{-v}\,v^{p(\lambda+1)}\,dv\\
&= \text{const}\,(p\ln r^{-1})^{-p(\lambda+1)-1}\,\Gamma(p(\lambda+1)+1)\\
&\le \text{const}\,(1-r)^{-p(\lambda+1)-1}. \qquad (3.4)
\end{aligned}$$

Note that the estimate (3.4) holds for all $p \ge 1$. Since

$$\left(\lambda+1-\frac{1}{p^2}\right)p - p(\lambda+1) - 1 = -(1+\frac{1}{p}),$$

from (3.3) and (3.4) it follows that

$$\begin{aligned}\|T_a^{(\lambda)}\|_{p,\lambda} &\le \text{const}\left(\int_0^\infty |a(\sqrt{r})|(1-r)^{-(1+\frac{1}{p})}\,dr\right)^{\frac{1}{q}}\\ &\times \left(\int_0^\infty |a(\sqrt{r})|(1-r)^{-(1+\frac{1}{p})}\,dr\right)^{\frac{1}{p}}\\ &= \text{const}\int_0^\infty |a(\sqrt{r})|(1-r)^{-(1+\frac{1}{p})}\,dr.\end{aligned}$$

Thus according to (3.1) with $j=0$ we get $T_a^{(\lambda)} \in K_p(\lambda)$.

If $p=1$ then

$$\begin{aligned}\|T_a^{(\lambda)}\|_{1,\lambda} &\le \sum_{n=0}^\infty \frac{1}{B(n+1,\lambda+1)}\int_0^1 |a(\sqrt{r})|(1-r)^\lambda r^n dr\\ &\le \int_0^1 |a(\sqrt{r})|(1-r)^\lambda \left(\sum_{n=0}^\infty \frac{r^n}{B(n+1,\lambda+1)}\right) dr.\end{aligned}$$

Since

$$\sum_{n=0}^\infty \frac{r^n}{B(n+1,\lambda+1)} = (\lambda+1)(1-r)^{-(\lambda+2)} \tag{3.5}$$

we have

$$\|T_a^{(\lambda)}\|_{1,\lambda} \le (\lambda+1)\int_0^1 |a(\sqrt{r})|(1-r)^{-2}dr,$$

and thus $T_a^{(\lambda)} \in K_1(\lambda)$ according to (3.1) with $j=0$.

Let now $p>1$ and let the condition (3.2) hold with $j=0$. Applying the Hölder inequality in an another way and using (3.4) we have

$$\begin{aligned}\|T_a^{(\lambda)}\|_{p,\lambda} &= \left(\sum_{n=0}^\infty \left|\frac{1}{B(n+1,\lambda+1)}\int_0^1 \left(a(\sqrt{r})(1-r)^{(\lambda+\frac{1-\sigma}{q})}\, r^n\right)\right.\right.\\ &\times \left.\left.\left((1-r)^{-\frac{1-\sigma}{q}}\right) dr\right|^p\right)^{\frac{1}{p}}\\ &\le \text{const}\left(\int_0^1 (1-r)^{-(1-\sigma)}dr\right)^{\frac{1}{q}}\\ &\times \left(\int_0^1 |a(\sqrt{r})|^p(1-r)^{(\lambda+\frac{1-\sigma}{q})p-(\lambda+1)p-1}dr\right)^{\frac{1}{p}}.\end{aligned}$$

Taking $\sigma = \frac{\varepsilon}{p-1} > 0$ we have

$$\|T_a^{(\lambda)}\|_{p,\lambda} \le \text{const}\left(\int_0^1 |a(\sqrt{r})|^p(1-r)^{-(2-\varepsilon)}dr\right)^{\frac{1}{p}},$$

and thus $T_a^{(\lambda)} \in K_p(\lambda)$ according to (3.2).

The case $j > 0$ is considered by integrating by parts (as in the proof of Theorem 2.1) and by repeating the above arguments. □

The next theorem shows that for a nonnegative symbol or for a symbol having any nonnegative mean $B_a^{(j)}(r)$, condition (3.1) is necessary as well for trace class operators, i.e., for $p = 1$.

Theorem 3.2. *Given a symbol $a(r)$, let for some $j \in \mathbb{Z}_+$ $B_a^{(j)}(r) \geq 0$ a.e. and $T_a^{(\lambda)} \in K_1(\lambda)$. Then*

$$\int_0^1 B_a^{(j)}(r)\,(1-r)^{-(2+j)}\,dr < \infty. \tag{3.6}$$

Proof. Let first $j = 0$, then according to (3.5) we have

$$\|T_a^{(\lambda)}\|_{1,\lambda} = (\lambda+1)\int_0^1 a(\sqrt{r})\,(1-r)^{-2}\,dr < \infty,$$

and the condition (3.6) holds.

Let now $j = 1$ and thus $B_a^{(1)}(r) \geq 0$ a.e. Integrating by parts we have

$$\begin{aligned}\|T_a^{(\lambda)}\|_{1,\lambda} = {} & \sum_{n=1}^{\infty}\frac{1}{B(n+1,\lambda+1)}\int_0^1 B_a^{(1)}(r)\,(\lambda(1-r)^{\lambda-1}r^n + n(1-r)^{\lambda}r^{n-1})dr \\ & + \int_0^1 \lambda(1-r)^{\lambda-1}B_a^{(1)}(r)dr + B_a^{(1)}(0).\end{aligned}$$

Using (3.5) and the representation

$$\sum_{n=1}^{\infty}\frac{nr^{n-1}}{B(n+1,\lambda+1)} = (\lambda+1)(\lambda+2)(1-r)^{-(\lambda+3)}$$

we get (3.6):

$$\begin{aligned}\|T_a^{(\lambda)}\|_{1,\lambda} &= \lambda(\lambda+1)\int_0^1 B_a^{(1)}(r)\,(1-r)^{-3}\,dr \\ &+ (\lambda+1)(\lambda+2)\int_0^1 B_a^{(1)}(r)\,(1-r)^{-3}\,dr + B_a^{(1)}(0) \\ &= 2(\lambda+1)^2\int_0^1 B_a^{(1)}(r)\,(1-r)^{-3}\,dr + B_a^{(1)}(0).\end{aligned}$$

The cases $j > 1$ are considered analogously. □

We give now several examples to the above theorems.

Example 3.3. For the symbol $a(r) = (1-r^2)^{\alpha}$ we have

$$\gamma_{a,\lambda}(n) = \frac{1}{B(n+1,\lambda+1)}\int_0^1 (1-r)^{\lambda+\alpha}\,r^n\,dr = \frac{B(n+1,\lambda+\alpha+1)}{B(n+1,\lambda+1)}.$$

The asymptotic representation (2.9) implies

$$\gamma_{a,\lambda}(n) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)}\,(n+1)^{-\alpha}\,\left(1+O\left(\frac{1}{n+1}\right)\right).$$

Thus there exists C_1, $C_2>0$ such that

$$C_1\left(\sum_{n=0}^{\infty}(n+1)^{-\alpha p}\right)^{\frac{1}{p}} \le \|T_a^{(\lambda)}\|_{p,\lambda} \le C_2\left(\sum_{n=0}^{\infty}(n+1)^{-\alpha p}\right)^{\frac{1}{p}},$$

and $T_a^{(\lambda)} \in K_p(\lambda)$ if and only if $\alpha \in (\frac{1}{p},\infty)$.

Note, that for $\alpha \in (\frac{1}{p},\infty)$ the above symbol satisfies the both conditions (3.1) and (3.2).

Example 3.4. Unbounded increasing sequence.
Consider the following nonnegative symbol

$$a(\sqrt{r}) = \begin{cases} a_k, & r \in I_k = [r_k, r_k+\varepsilon_k], \qquad k \in \mathbb{N}, \\ 0 & r \in [0,1]\setminus \cup_{k=1}^{\infty} I_k, \end{cases}$$

where $\{r_k\}$ is a positive increasing sequence tending to 1, $r_k+\varepsilon_k < r_{k+1}$, and a_k, $\varepsilon_k>0$.

The conditions (3.1) and (3.2) for $j=0$ for such a symbol are as follows

$$\begin{aligned} p\sum_{k=1}^{\infty} a_k\left((1-(r_k+\varepsilon_k))^{-\frac{1}{p}} - (1-r_k)^{-\frac{1}{p}}\right) &< \infty, \\ (1-\varepsilon)^{-1}\sum_{k=1}^{\infty} a_k^p\left((1-(r_k+\varepsilon_k))^{-(1-\varepsilon)} - (1-r_k)^{-(1-\varepsilon)}\right) &< \infty. \end{aligned}$$

Setting

$$a_k = k^a, \quad a>0, \quad r_k = 1-\frac{1}{k}, \quad \varepsilon_k = \frac{k^{-b}}{2}, \quad b > 2+\frac{1}{p},$$

the above conditions are equivalent to

$$\sum_{k=1}^{\infty} k^a \cdot k^{\frac{1}{p}-b+1} < \infty, \tag{3.7}$$

$$\sum_{k=1}^{\infty} k^{ap} \cdot k^{(1-\varepsilon)-b+1} < \infty. \tag{3.8}$$

Thus (3.7) holds if

$$a < b-2-\frac{1}{p},$$

while (3.8) holds if

$$a < \frac{b-3}{p},$$

recalling that $\varepsilon>0$ can be chosen arbitrary small.

Since for our choice of b we have (for $p > 1$)

$$b - 2 - \frac{1}{p} > \frac{b-3}{p},$$

thus the first condition gives a bigger region for $T_a^{(\lambda)} \in K_p(\lambda)$, and thus the first condition of Theorem 3.1 is better adapted for increasing sequences $\{a_k\}$ that the second one.

Example 3.5. Decreasing sequence.
Consider the symbol from the previous example, but setting now

$$a_k = k^{-a}, \quad a > 0, \quad r_k = 1 - \frac{1}{k}, \quad \varepsilon_k = \frac{k^{-b}}{2}, \quad b \in [2, 2 + \frac{1}{p}].$$

In this case after a short calculation we come to the following conditions:

$$\begin{array}{rcl} 2 + \frac{1}{p} - b & > & a, \\ \frac{3-b}{p} & > & a. \end{array}$$

For the current choice of b we have (for $p > 1$)

$$\frac{3-b}{p} > 2 + \frac{1}{p} - b,$$

and thus now the second condition gives a bigger region for $T_a^{(\lambda)} \in K_p(\lambda)$; that is, the second condition of Theorem 3.1 is better adapted for decreasing sequences $\{a_k\}$ than the first one.

In the three previous examples the symbols were nonnegative. Consider now

Example 3.6. Unbounded oscillating symbol.
Let

$$a(r) = (1-r^2)^{-\beta} \sin(1-r^2)^{-\alpha}, \qquad \alpha > 0, \ \ 0 < \beta < 1.$$

We have the following asymptotic representation in a neighborhood of $r = 1$ (see [8], (3.4))

$$B_a^{(1)}(r) = \frac{\cos(1-r)^{-\alpha}}{\alpha} (1-r)^{\alpha-\beta+1} + O((1-r)^{2\alpha-\beta+1}). \tag{3.9}$$

Thus by condition (3.1) for $j = 1$ of Theorem 3.1 we have $T_a^{(\lambda)} \in K_p(\lambda)$ if

$$(\alpha - \beta + 1) - (1 + 1 + \frac{1}{p}) > -1 \qquad \text{or} \qquad \alpha - \beta > \frac{1}{p}.$$

Setting $\beta_1 = \alpha - \beta + 1$, analogously to (3.9) we can get the following representation

$$B_a^{(2)}(r) = -\frac{\sin(1-r)^{-\alpha}}{\alpha^2} (1-r)^{\alpha+\beta_1+1} + O((1-r)^{2\alpha+\beta_1+1}).$$

Apply now condition (3.7) for $j = 2$; we have that $T_a^{(\lambda)} \in K_p(\lambda)$ if

$$(\alpha + \beta_1 + 1) - (1 + 2 + \frac{1}{p}) > -1 \qquad \text{or} \qquad 2\alpha - \beta > \frac{1}{p}\,.$$

In the same manner for any $j \in \mathbb{N}$ using the asymptotic representation of the corresponding mean $B^{(j)}(r)$ we can get that $T_a^{(\lambda)} \in K_p(\lambda)$ if

$$j\alpha - \beta > \frac{1}{p}\,. \tag{3.10}$$

Since for any $\alpha > 0$ and $0 < \beta < 1$ there exists $j \in \mathbb{N}$ such that (3.10) holds, the operator $T_a^{(\lambda)}$ always belongs to $K_p(\lambda)$, for any $\lambda \geq 0$, $p \geq 1$, $\alpha > 0$, and $0 < \beta < 1$.

Theorem 3.7. *Let $T_a^{(\lambda_0)} \in K_p(\lambda_0)$ for some $\lambda_0 > 0$, and let $1 \leq p < \infty$. Then for all $\lambda \in [0, \lambda_0]$ we have*

$$T_a^{(\lambda)} \in K_p(\lambda).$$

Proof. Using the representation (2.8) we have

$$\begin{aligned} \|T_a^{(\lambda)}\|_{p,\lambda} &= \Bigg(\sum_{n=1}^{\infty} \Bigg| \frac{\sin \pi(\lambda_0 - \lambda)}{\pi} \sum_{j=0}^{\infty} \frac{B(j - (\lambda - \lambda_0), 1 + (\lambda - \lambda_0))}{B(n+1, \lambda+1)} \\ &\times \quad B(n+j+1, \lambda_0+1)\gamma_{a\lambda_0}(n+j) \Bigg|^p \Bigg)^{1/p}. \end{aligned}$$

Then by (2.9)

$$\begin{aligned} \|T_a^{(\lambda)}\|_{p,\lambda} &\leq \text{const} \left(\sum_{n=1}^{\infty} \left| \sum_{j=0}^{\infty} \frac{n^{\lambda+1}\,\gamma_{a,\lambda_0}(n+j)}{(n+j+1)^{\lambda_0+1}(j+1)^{1-(\lambda_0-\lambda)}} \right|^p \right)^{1/p} \\ &= \text{const} \left(\sum_{n=1}^{\infty} \left| \sum_{\nu=0}^{\infty} \sum_{\nu \leq \frac{j+1}{n} < \nu+1} \frac{\gamma_{a,\lambda_0}(n+j)}{n\left(1 + \frac{j+1}{n}\right)^{\lambda+1} \left(\frac{j+1}{n}\right)^{1-(\lambda_0-\lambda)}} \right|^p \right)^{1/p} \\ &=: \text{const} \left(\sum_{n=1}^{\infty} \left| \sum_{\nu=0}^{\infty} \theta_\nu(n) \right|^p \right)^{1/p}. \end{aligned}$$

Using the Minkowski inequality we have

$$\|T_a^{(\lambda)}\|_{p,\lambda} \leq \text{const} \sum_{\nu=0}^{\infty} \left(\sum_{n=1}^{\infty} |\theta_\nu(n)|^p \right)^{1/p}.$$

It is easy to see that

$$|\theta_\nu(n)| \leq \frac{1}{(1+\nu)^{\lambda+1}\,\nu^{1-(\lambda_0-\lambda)}} \sum_{\nu \leq \frac{j+1}{n} < \nu+1} \frac{|\gamma_{a,\lambda_0}(n+j)|}{n}.$$

for each $\nu = 1, 2, \ldots$.

Suppose first that $p > 1$. According to the Hölder inequality

$$|\theta_\nu(n)|^p \leq \frac{1}{(1+\nu)^{p(\lambda+1)}\,\nu^{p(1-(\lambda_0-\lambda))}} \left(\frac{n}{n^q}\right)^{p/q} \sum_{\nu \leq \frac{j+1}{n} < \nu+1} |\gamma_{a,\lambda_0}(n+j)|^p,$$

where $1/p + 1/q = 1$. So we have

$$|\theta_\nu(n)|^p \leq \frac{\text{const}}{(1+\nu)^{(\lambda+1)p}\,\nu^{(1-(\lambda-\lambda_0))p}} \sum_{\nu \leq \frac{j+1}{n} < \nu+1} \frac{|\gamma_{a,\lambda_0}(n+j)|^p}{n},$$

for $\nu = 1, 2, \ldots$. Thus

$$\begin{aligned} \|T_a^{(\lambda)}\|_{p,\lambda} &\leq \text{const}\left[\left(\sum_{n=1}^{\infty} |\theta_0(n)|^p\right)^{1/p}\right. \\ &+ \left.\sum_{\nu=1}^{\infty} \frac{1}{(1+\nu)^{\lambda+1}\,\nu^{1-(\lambda-\lambda_0)}} \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\nu \leq \frac{j+1}{n} < \nu+1} |\gamma_{a,\lambda_0}(n+j)|^p\right)^{1/p}\right] \end{aligned}$$

Consider first the series

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\nu \leq \frac{j+1}{n} < \nu+1} |\gamma_{a,\lambda_0}(n+j)|^p &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\nu+1 \leq \frac{m+1}{n} < \nu+2} |\gamma_{a,\lambda_0}(m)|^p \\ &= \sum_{m=\nu}^{\infty} |\gamma_{a,\lambda_0}(m)|^p \sum_{\frac{m+1}{\nu+2} < n \leq \frac{m+1}{\nu+1}} \frac{1}{n} \\ &\leq \text{const} \sum_{m=\nu}^{\infty} |\gamma_{a,\lambda_0}(m)|^p \int_{\frac{m+1}{\nu+2}}^{\frac{m+1}{\nu+1}} \frac{du}{u} \\ &= \text{const} \sum_{m=\nu}^{\infty} |\gamma_{a,\lambda_0}(m)|^p \left(\ln\frac{m+1}{\nu+1} - \ln\frac{m+1}{\nu+2}\right) \\ &= \text{const}\ \ln\frac{\nu+2}{\nu+1} \cdot \|T_a^{(\lambda_0)}\|_{p,\lambda_0}^p. \end{aligned}$$

Thus

$$\begin{aligned} \|T_a^{(\lambda)}\|p,\lambda &\leq \text{const}\left[\left(\sum_{n=1}^{\infty} n^{-(\lambda_0-\lambda)p} \left|\sum_{\frac{j+1}{n} < 1} \frac{\gamma_{a,\lambda_0}(n+j)}{(1+\frac{j+1}{n})^{\lambda+1}\,(j+1)^{1-(\lambda_0-\lambda)}}\right|^p\right)^{1/p}\right. \\ &+ \left.\|T_a^{(\lambda_0)}\|_{p,\lambda_0} \sum_{\nu=1}^{\infty} \frac{\ln^{1/p}(1+\frac{1}{\nu+1})}{(1+\nu)^{\lambda+1}\,\nu^{1-(\lambda-\lambda_0)}}\right]. \end{aligned} \tag{3.11}$$

The second term of (3.11) is bounded as

$$\lambda + 1 + (1 - (\lambda - \lambda_0)) + \frac{1}{p} = 2 + \lambda_0 + \frac{1}{p} > 1.$$

Applying the Hölder inequality to the first term of (3.11) we get

$$\begin{aligned}
I_1 &= \left(\sum_{n=1}^{\infty} n^{-(\lambda_0-\lambda)p} \left| \sum_{j+1<n} \frac{\gamma_{a,\lambda_0}(n+j)}{\left(1+\frac{j+1}{n}\right)^{\lambda+1}(j+1)^{1-(\lambda_0-\lambda)}} \right|^p \right)^{1/p} \\
&\le \left(\sum_{n=1}^{\infty} n^{-(\lambda_0-\lambda)p} \left| \sum_{j+1<n} \left(\frac{\gamma_{a,\lambda_0}(n+j)}{(j+1)^{\frac{1-(\lambda_0-\lambda)}{p}}}\right) \cdot \frac{1}{(j+1)^{\frac{1-(\lambda_0-\lambda)}{q}}} \right|^p \right)^{1/p} \\
&\le \left(\sum_{n=1}^{\infty} n^{-(\lambda_0-\lambda)p} \sum_{j+1<n} \frac{|\gamma_{a,\lambda_0}(n+j)|^p}{(j+1)^{1-(\lambda_0-\lambda)}} \cdot \left(\sum_{j+1<n} \frac{1}{(j+1)^{1-(\lambda_0-\lambda)}}\right)^{p/q} \right)^{1/p} \\
&\le \text{const} \left(\sum_{n=1}^{\infty} n^{-(\lambda_0-\lambda)p}\, n^{(\lambda_0-\lambda)\frac{p}{q}} \sum_{j+1<n} \frac{|\gamma_{a,\lambda_0}(n+j)|^p}{(j+1)^{1-(\lambda_0-\lambda)}}\right)^{1/p} \\
&= \text{const} \left(\sum_{n=1}^{\infty} n^{-(\lambda_0-\lambda)} \sum_{n+1\le m+1<2n} \frac{|\gamma_{a,\lambda_0}(m)|^p}{(m-n+1)^{1-(\lambda_0-\lambda)}}\right)^{1/p} \\
&\le \text{const} \left(\sum_{m=1}^{\infty} |\gamma_{a,\lambda_0}(m)|^p \sum_{\frac{m+1}{2}<n\le m} \frac{n^{-(\lambda_0-\lambda)}}{(m-n+1)^{1-(\lambda_0-\lambda)}}\right)^{1/p} \\
&\le \text{const} \left(\sum_{m=1}^{\infty} |\gamma_{a,\lambda_0}(m)|^p \int_{\frac{m+1}{2}}^{m+1} \frac{u^{-(\lambda_0-\lambda)}\,du}{(m+1-u)^{1-(\lambda_0-\lambda)}}\right)^{1/p}.
\end{aligned}$$

Changing $u = (m+1)v$ in the last integral we have

$$I_1 \le \text{const} \left(\sum_{m=1}^{\infty} |\gamma_{a,\lambda_0}(m)|^p \int_{\frac{1}{2}}^{1} \frac{v^{-(\lambda_0-\lambda)}\,dv}{(1-v)^{1-(\lambda_0-\lambda)}}\right)^{1/p}.$$

Since $1-(\lambda_0-\lambda) < 1$, the last integral exists, and the first summand in (3.11) is estimated as follows

$$I_1 \le \text{const}\, \|T_a^{(\lambda_0)}\|_{p,\lambda_0}.$$

Thus (for $p > 1$) we have

$$\|T_a^{(\lambda)}\|_{p,\lambda} \le \text{const}\, \|T_a^{(\lambda_0)}\|_{p,\lambda_0}.$$

The case $p = 1$ is considered analogously. □

Note that the above result admits the following reformulation. Given a radial symbol $a(r)$, let $T_a^{(\lambda_0)} \in K_{p_0}(\lambda_0)$. Then $T_a^{(\lambda)} \in K_p(\lambda)$ for all $(p,\lambda) \in [p_0,\infty) \times [0,\lambda_0]$.

4. Spectra of Toeplitz operators, continuous symbols

The following notion will be useful in the sequel. Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point (normally $E = [0, +\infty)$), and let for each $\lambda \in E$ there is a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,
(ii) $\lim_{n\to\infty} \lambda_n = +\infty$,
(iii) $z = \lim_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of a family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

For the case when E is a discrete set with a unique limit point at infinity, the above notion coincides with the partial limiting set introduced in [9], Section 3.1.1. Following the arguments of Proposition 3.5 in [9] one can show that

$$M_\infty = \bigcap_\lambda \operatorname{clos}\left(\bigcup_{\mu\geq\lambda} M_\mu\right).$$

Note that obviously

$$\lim_{\lambda\to+\infty} M_\lambda = \lim_{\lambda\to+\infty} \overline{M}_\lambda = M_\infty.$$

The *a priori* spectral information for Toeplitz operators with L_∞-symbols is given, for example, in [4], and says that for each $a \in L_\infty(\mathbb{D})$ and each $\lambda > -1$

$$\operatorname{sp} T_a^{(\lambda)} \subset \operatorname{conv}(\text{ess-Range}\, a). \tag{4.1}$$

For a radial symbol $a = a(r)$ the Toeplitz operator $T_a^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, where the sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ is given by (2.1). Thus we have obviously

$$\operatorname{sp} T_a^{(\lambda)} = \overline{M_\lambda(a)},$$

where $M_\lambda(a) = \operatorname{Range} \gamma_{a,\lambda}$.

An interesting question here is to study the limit behavior of $\operatorname{sp} T_a^{(\lambda)}$ when $\lambda \to \infty$.

Theorem 4.1. *Let $a = a(r) \in C[0,1]$. Then*

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = \lim_{\lambda\to+\infty} M_\lambda(a) = \operatorname{Range} a. \tag{4.2}$$

Obviously the set $\operatorname{Range} a$ coincides with the spectrum $\operatorname{sp} aI$ of the operator of multiplication by $a = a(r)$ acting, say, on any of $L_2(\mathbb{D}, d\mu_\lambda)$, thus another form of (4.2) is

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{sp} aI.$$

Proof. We find the asymptotics of (2.1) when $\lambda \to +\infty$ using the Laplace method. Introduce a large parameter $L = \sqrt{\lambda^2 + n^2}$, and represent (2.1) in the form

$$\gamma_{a,\lambda}(n) = \frac{1}{\mathrm{B}(n+1, \lambda+1)} \int_0^1 a(\sqrt{r})\, e^{-LS(r,\varphi)}\, dr, \tag{4.3}$$

where $S(r,\varphi) = \sin\varphi \cdot \ln(1-r)^{-1} + \cos\varphi \cdot \ln r^{-1}$ and $\sin\varphi = \frac{\lambda}{L}$, $\cos\varphi = \frac{n}{L}$ with $\varphi \in [0, \frac{\pi}{2})$.

Find the point of maximum of $S(r,\varphi)$, we have

$$S'_r(r,\varphi) = \frac{\sin\varphi}{1-r} - \frac{\cos\varphi}{r},$$

and obviously $S'_r(r,\varphi) = 0$ for

$$r_0 = \frac{\cos\varphi}{\sin\varphi + \cos\varphi} \in [0,1],$$

moreover for $\varphi = 0$ $r_0 = 1$ and for $\varphi = \frac{\pi}{2}$ $r_0 = 0$.

Rewrite now (4.3) in the form

$$\begin{aligned} \gamma_{a,\lambda}(n) &= \frac{1}{\mathrm{B}(n+1,\lambda+1)} \left(\int_0^1 a(\sqrt{r_0})\, e^{-LS(r,\varphi)}\, dr \right. \\ &+ \int_{U(r_0)} (a(\sqrt{r}) - a(\sqrt{r_0}))\, e^{-LS(r,\varphi)}\, dr \\ &+ \left. \int_{[0,1]\setminus U(r_0)} (a(\sqrt{r}) - a(\sqrt{r_0}))\, e^{-LS(r,\varphi)}\, dr \right), \end{aligned}$$

where $U(r_0)$ is a neighborhood of r_0 such that for a sufficiently small ε

$$\sup_{r\in U(r_0)} |a(\sqrt{r}) - a(\sqrt{r_0})| < \varepsilon.$$

We have

$$\int_0^1 a(\sqrt{r_0})\, e^{-LS(r,\varphi)}\, dr = a(\sqrt{r_0})\, \mathrm{B}(n+1, \lambda+1).$$

Further

$$\left| \int_{U(r_0)} (a(\sqrt{r}) - a(\sqrt{r_0}))\, e^{-LS(r,\varphi)}\, dr \right| \le \varepsilon \int_{U(r_0)} e^{-LS(r,\varphi)}\, dr \le \varepsilon\, \mathrm{B}(n+1,\lambda+1).$$

And, finally,

$$\begin{aligned} & \left| \int_{[0,1]\setminus U(r_0)} (a(\sqrt{r}) - a(\sqrt{r_0}))\, e^{-LS(r,\varphi)}\, dr \right| \\ \le\ & 2 \sup_{r\in[0,1]} |a(\sqrt{r})| \int_{[0,1]\setminus U(r_0)} e^{-LS(r,\varphi)}\, dr \\ \le\ & \left(2 \sup_{r\in[0,1]} |a(\sqrt{r})|\, e^{-L\delta(\varepsilon)} \right) \mathrm{B}(n+1,\lambda+1), \end{aligned}$$

where $\delta(\varepsilon) > 0$ can be taken independently on $r_0 \in [0, 1]$.

Since ε can be taken arbitrary small uniformly on $r_0 \in [0, 1]$, we have the following representation

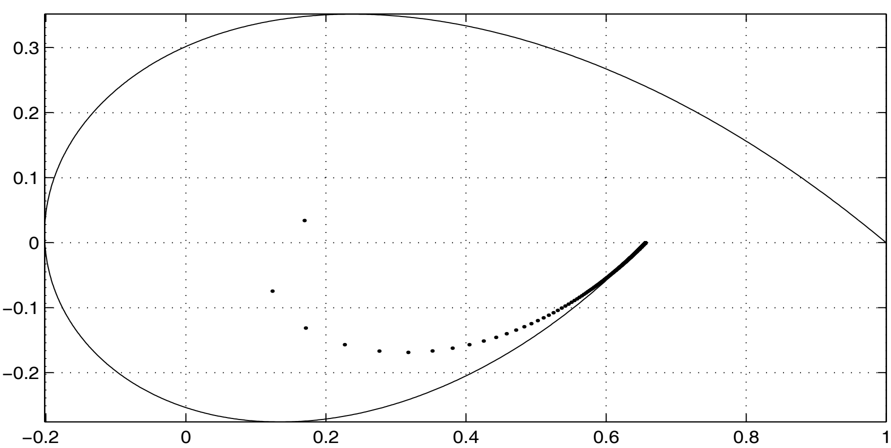
$$\gamma_{a,\lambda}(n) = a(\sqrt{r_0})(1 + \alpha(L)), \tag{4.4}$$

where $\lim_{L\to\infty} \alpha(L) = 0$ uniformly on $r_0 \in [0, 1]$, or $\varphi \in [0, \frac{\pi}{2}]$.

Finally, the last representation guaranties the statement of the theorem. □

Recall that for a continuous symbol $a(r)$ and for each fixed λ the spectrum $\operatorname{sp} T_a^{(\lambda)}$ coincides with the closure of the set $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$, i.e., is a discrete set with the unique limit point $a(1)$, and, in general, seems to have no any strict connection with the range of $a(r)$. The definite tendency in the behavior of elements of the sequence $\gamma_{a,\lambda}$ starts appearing as λ tends to ∞, and the limit set $M_\infty(a)$ of those sequences coincides with the range of the initial symbol $a(r)$.

We illustrate this effect on the symbol $a(r) = (1 + (0.9i - 1)r^2)^4$ for three values of λ: $\lambda = 0$, $\lambda = 5$, and $\lambda = 100$.



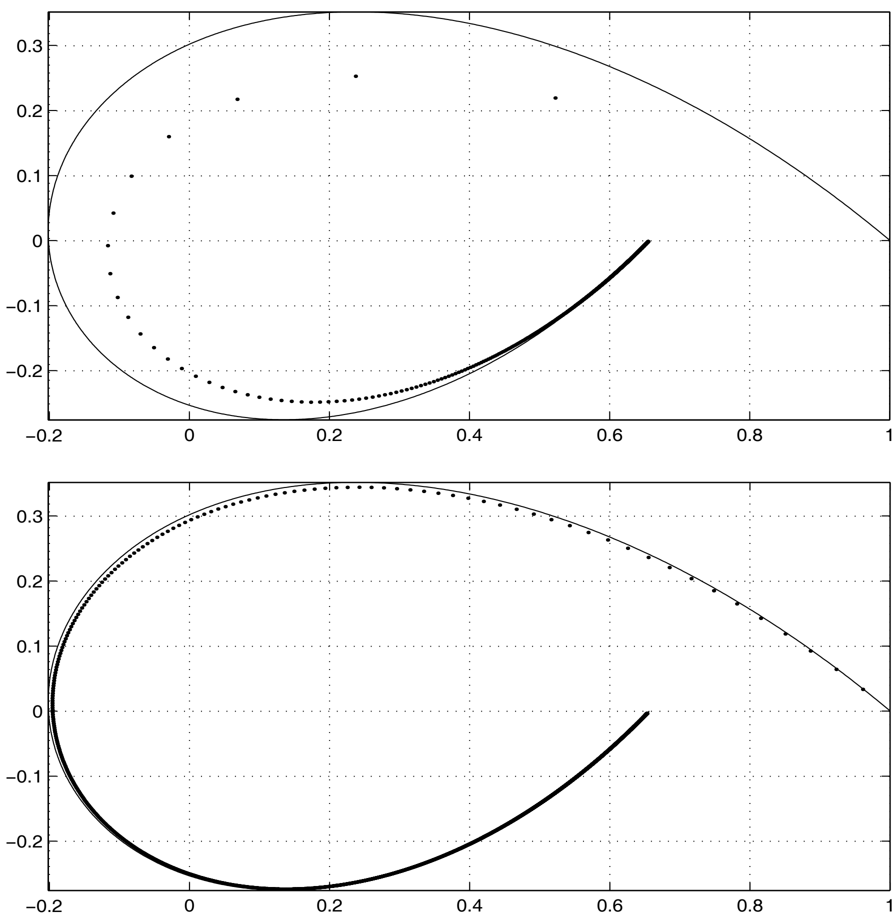


FIGURE 1. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ for $\lambda = 0$, $\lambda = 5$, and $\lambda = 100$.

5. Spectra of Toeplitz operators, piecewise continuous symbols

Let $b(r) = a(\sqrt{r})$ be a piecewise continuous function having jumps at a finite set of points,

$$0 < r_1 < r_2 < ... < r_m < 1.$$

Introduce the sets

$$J_j(a) = \{z \in \mathbb{C} : z = a(\sqrt{r}), r \in [r_j + 0, r_{j+1} - 0]\},$$

where $j = 0, 1, \ldots, m$, $r_0 = 0$ and $r_{m+1} = 1$, and $I_j(a)$ being the straight line segment with the endpoints $a(\sqrt{r_j - 0})$ and $a(\sqrt{r_j + 0})$, for all $j = 0, 1, \ldots, m$.

Introduce now

$$\widetilde{R}(a) = \left(\bigcup_{j=0}^{m} J_j(a) \right) \cup \left(\bigcup_{j=1}^{m} I_j(a) \right),$$

that is,

$$\widetilde{R}(a) = \operatorname{Range} a \cup \left(\bigcup_{j=1}^{m} I_j(a) \right),$$

Theorem 5.1. *Let $b(r) = a(\sqrt{r})$ be a piecewise continuous function as above. Then*

$$\lim_{\lambda \to +\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

Proof. We use the Laplace method as in Theorem 4.1. For any $\varepsilon > 0$ we take $\delta > 0$ such that for each interval $I \subset (r_j, r_{j+1})$ with length less than δ , $j = 1, 2, \ldots, m$, the following holds

$$\sup_{s_1, s_2 \in I} |a(\sqrt{s_1}) - a(\sqrt{s_2})| < \varepsilon.$$

Suppose first that the maximum point $s_0 = \frac{\cos\varphi}{\sin\varphi + \cos\varphi}$ satisfies the condition

$$\sup_{j=1,2,\ldots,m} |s_0 - r_j| \geq \delta.$$

We have

$$\begin{aligned} \gamma_{a,\lambda}(n) &= a(\sqrt{s_0}) + \frac{1}{\mathrm{B}(n+1, \lambda+1)} \int_{s_0-\delta}^{s_0+\delta} (a(\sqrt{r}) - a(\sqrt{s_0}))\, e^{-LS(r,\varphi)} dr \\ &+ \frac{1}{\mathrm{B}(n+1, \lambda+1)} \int_{[0,1]\setminus(s_0-\delta, s_0+\delta)} (a(\sqrt{r}) - a(\sqrt{s_0}))\, e^{-LS(r,\varphi)} dr \\ &= a(\sqrt{s_0}) + O(\varepsilon) + O(e^{-\sigma L}), \end{aligned} \tag{5.1}$$

where $\sigma = \min_{[0,1]\setminus(s_0-\delta, s_0+\delta)} (S(r,\varphi) - S(s_0,\varphi))$. Thus, varying φ and δ, we have that

$$J_j(a) \subset M_\infty(a), \qquad j = 0, 1, \ldots, m.$$

Now suppose that there exists j such that $|s_0 - r_j| < \delta$. The we have

$$\begin{aligned}
\gamma_{a,\lambda}(n) &= \frac{1}{\mathrm{B}(n+1,\lambda+1)}\left(a(\sqrt{r_j-0})\int_{s_0-\delta}^{r_j} e^{-LS(r,\varphi)}dr\right. \\
&+ \left. a(\sqrt{r_j+0})\int_{r_j}^{s_0+\delta} e^{-LS(r,\varphi)}dr\right) \\
&+ \frac{1}{\mathrm{B}(n+1,\lambda+1)}\left(\int_{s_0-\delta}^{r_j}(a(\sqrt{r})-a(\sqrt{r_j-0}))\,e^{-LS(r,\varphi)}dr\right. \\
&+ \int_{r_j}^{s_0+\delta}(a(\sqrt{r})-a(\sqrt{r_j+0}))\,e^{-LS(r,\varphi)}dr \\
&+ \left.\int_{[0,1]\setminus(s_0-\delta,s_0+\delta)} a(\sqrt{r})\,e^{-LS(r,\varphi)}dr\right) \\
&= \frac{1}{\mathrm{B}(n+1,\lambda+1)}\left(a(\sqrt{r_j-0})\int_{s_0-\delta}^{r_j} e^{-LS(r,\varphi)}dr\right. \\
&+ \left. a(\sqrt{r_j+0})\int_{r_j}^{s_0+\delta} e^{-LS(r,\varphi)}dr\right) + O(\varepsilon) + O(e^{-L\sigma}). \qquad (5.2)
\end{aligned}$$

Taking δ small enough we can suppose that

$$\frac{r_1}{2} < s_0 < \frac{1+r_n}{2}\,.$$

Thus the functions $(S''(s_0,\varphi))^{\pm1}$ are uniformly bounded (on s_0), and then the following asymptotic calculations are uniform on s_0

$$\begin{aligned}
\mathrm{B}(n+1,\lambda+1) &= \int_0^1 e^{-LS(r,\varphi)}dr = e^{-LS(s_0,\varphi)}\int_0^1 e^{-L\frac{S''(s_0,\varphi)(r-s_0)^2}{2}}dr\,(1+o(1)) \\
&= e^{-LS(s_0,\varphi)}\int_{-s_0}^{1-s_0} e^{-L\frac{S''(s_0,\varphi)u^2}{2}}du\,(1+o(1)) \\
&= \sqrt{\frac{2}{S''(s_0,\varphi)}}\,\frac{e^{-LS(s_0,\varphi)}}{L^{\frac12}}\int_{-\infty}^{+\infty} e^{-v^2}dv\,(1+o(1)). \qquad (5.3)
\end{aligned}$$

Analogously,

$$\int_{r_j}^{s_0+\delta} e^{-LS(r,\varphi)}dr = \sqrt{\frac{2}{S''(s_0,\varphi)}}\,\frac{e^{-LS(s_0,\varphi)}}{L^{\frac12}}\int_{x_j}^{+\infty} e^{-v^2}dv\,(1+o(1)), \qquad (5.4)$$

and

$$\int_{s_0-\delta}^{r_j} e^{-LS(r,\varphi)}dr = \sqrt{\frac{2}{S''(s_0,\varphi)}}\,\frac{e^{-LS(s_0,\varphi)}}{L^{\frac12}}\int_{-\infty}^{x_j} e^{-v^2}dv\,(1+o(1)), \qquad (5.5)$$

where

$$x_j = (r_j - s_0)\left(\frac{LS''(s_0,\varphi)}{2}\right)^{\frac{1}{2}}. \tag{5.6}$$

Thus from (5.2) - (5.5) we have

$$\gamma_{a,\lambda}(n) = (a(\sqrt{r_j - 0})t + a(\sqrt{r_j + 0})\tau)\,(1 + o(1)) + O(\varepsilon) + O(e^{-L\sigma}), \tag{5.7}$$

where

$$t = \frac{\int_{-\infty}^{x_j} e^{-v^2}dv}{\int_{-\infty}^{+\infty} e^{-v^2}dv} \qquad \text{and} \qquad \tau = \frac{\int_{x_j}^{+\infty} e^{-v^2}dv}{\int_{-\infty}^{+\infty} e^{-v^2}dv}.$$

Now it is evident that $t, \tau \in [0,1]$ and $t + \tau = 1$. Moreover, according to (5.6) for each $x \in \mathbb{R}$ there exist s_0 and L such that

$$x_j = (r_j - s_0)\left(\frac{LS''(s_0,\varphi)}{2}\right)^{\frac{1}{2}} = x.$$

Thus t runs over the whole segment $[0,1]$, which implies $I_j(a) \subset M_\infty(a)$. Thus

$$\widetilde{R}(a) \subset M_\infty(a).$$

Representations (5.1) and (5.7) imply the inverse inclusion

$$\widetilde{R}(a) \supset M_\infty(a). \qquad \square$$

Remark 5.2. The appearance of line segments which connect the one-sided limit values at the points of discontinuity of symbols, is quite typical in the theory of Toeplitz operators with piecewise continuous symbols acting either on the Hardy, or on the Bergman space (see, for example, [6, 17]). We stress the principal difference between our case and the cases just mentioned. In the mentioned case of Toeplitz operators with piecewise continuous symbols the line segments appear in the *essential spectrum* of the Toeplitz operator. In our case any Toeplitz operator is a compact perturbation of a multiple of the identity, i.e., $T_{a(r)} = a(1)I + K$, and its essential spectrum consists of a single point $a(1)$ for all λ. For each fixed λ the spectrum of a Toeplitz operator coincides with the union of the *discrete set* (the sequence $\gamma_{a,\lambda}(n)$) with its limit point $a(1)$. The tendency of the line segment forming stars appearing for large values of λ, and the line segments themselves appear only in the *limit set of spectra*.

We illustrate this effect on the piecewise continuous symbol

$$a(r) = \begin{cases} e^{-i\pi r^2}, & r \in [0, 1/\sqrt{2}], \\ e^{i\pi r^2}, & r \in (1/\sqrt{2}, 1], \end{cases}$$

for λ taking the values 0, 4, 40, and 200.

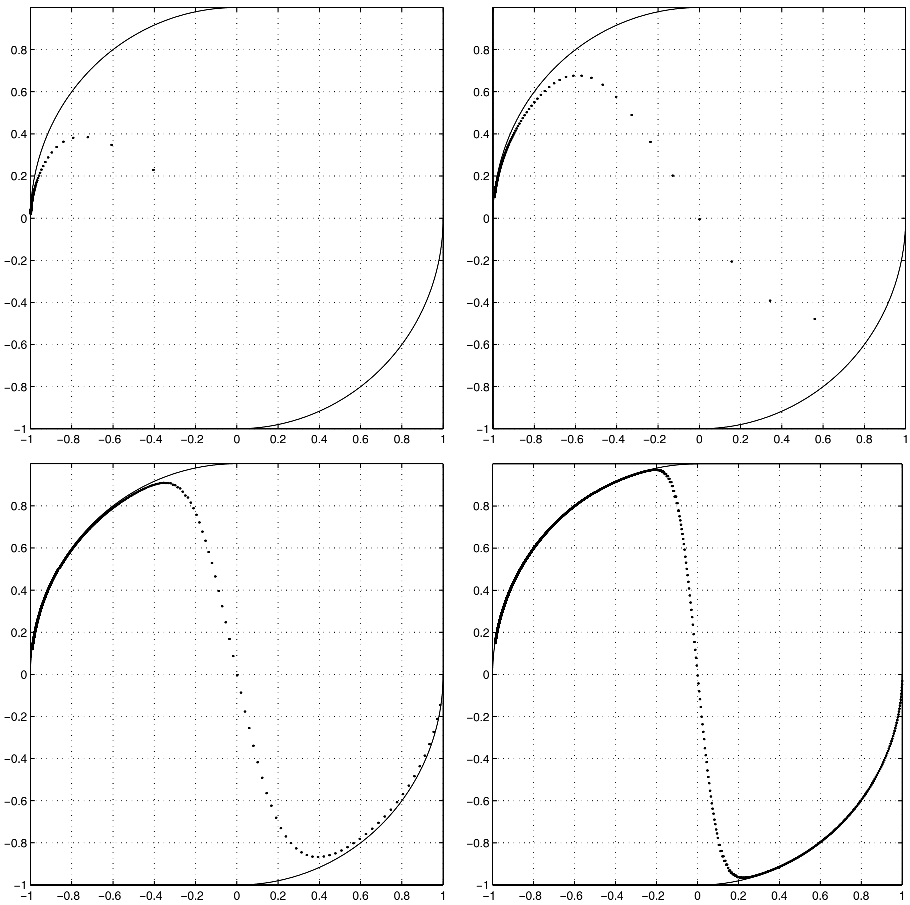


FIGURE 2. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ for $\lambda = 0$, $\lambda = 4$, $\lambda = 40$, and $\lambda = 200$.

For L_∞-symbols apart of the a priori information (4.1) we have obviously

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) \subset \operatorname{conv}(\text{ess-Range}\, a). \tag{5.8}$$

Let us give a number of examples illustrating possible interrelations between these sets.

EXAMPLE 5.3. Let $a = a(r) \in C[0, 1]$. Then according to Theorem 4.1

$$M_\infty(a) = \operatorname{Range} a \ (= \text{ess-Range}\, a).$$

EXAMPLE 5.4. Let

$$a(r) = \begin{cases} z_1, & r \in [0, \frac{1}{2}), \\ z_2, & r \in [\frac{1}{2}, 1], \end{cases}$$

where $z_1, z_2 \in \mathbb{C}$, and $z_1 \neq z_2$. Then according to Theorem 5.1 $M_\infty(a)$ coincides with the straight line segment $[z_1, z_2]$ joining the points z_1 and z_2, and thus

$$M_\infty(a) = \operatorname{conv}(\text{ess-Range}\, a) \ (= \operatorname{conv}(\operatorname{Range} a)).$$

Example 5.5. Let

$$a(r) = \begin{cases} z_1, & r \in [0, \frac{1}{4}), \\ z_2, & r \in [\frac{1}{4}, \frac{1}{2}), \\ z_3, & r \in [\frac{1}{2}, \frac{3}{4}), \\ z_4, & r \in [\frac{3}{4}, 1], \end{cases}$$

where z_1, z_2, z_3, z_4 are different points on the complex plane. By Theorem 5.1 we have

$$M_\infty(a) = [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_1] \cup [z_3, z_4],$$

and the set $M_\infty(a)$ is a part of the boundary of the convex hull of ess-Range $a =$ Range a:

$$M_\infty(a) \subset \partial \operatorname{conv}(\operatorname{Range} a).$$

If $z_4 = z_1$ then the set $M_\infty(a)$ coincides with the triangle having the vertices z_1, z_2, z_3, and

$$M_\infty(a) = \partial \operatorname{conv}(\operatorname{Range} a).$$

Example 5.6. Let $\{r_j\}_{j\in\mathbb{Z}_+}$ be a sequence of increasing positive numbers with $\lim_{j\to\infty} r_j = 1$. Define the symbol $a(\sqrt{r})$ as follows

$$a(\sqrt{r}) = \begin{cases} e^{i\theta_j}, & r \in [r_{2j}, r_{2j+1}), \\ -e^{i\theta_j}, & r \in [r_{2j+1}, r_{2j+2}), \qquad j = 0, 1, \ldots, \end{cases}$$

where $\{\theta_j\}_{j\in\mathbb{Z}_+}$ is a dense set in $[0, \pi]$.

As in the proof of Theorem 5.1, one can show that each diameter $[e^{i\theta_j}, -e^{i\theta_j}]$ of the unit disk $\mathbb{D}$ with endpoints $e^{i\theta_j}$ and $-e^{i\theta_j}$ belongs to $M_\infty(a)$, which implies $\overline{\mathbb{D}} \subset M_\infty(a)$.

We have that $\operatorname{clos} \operatorname{Range} a = \partial\, \mathbb{D} = \mathbb{T}$, thus finally

$$M_\infty(a) = \overline{\mathbb{D}} = \operatorname{conv}(\operatorname{Range} a).$$

In all examples above the set $M_\infty(a)$ is connected. But this is not so in general.

Example 5.7. Let $a(r) = r^{2i}$. Then Range $a = S^1$, and for a fixed λ the elements

$$\begin{aligned} \gamma_{a,\lambda}(n) &= \frac{1}{\mathrm{B}(n+1, \lambda+1)} \int_0^1 (1-r)^\lambda\, r^{n+i} dr \\ &= \frac{\mathrm{B}(n+1+i, \lambda+1)}{\mathrm{B}(n+1, \lambda+1)} = \frac{\Gamma(n+\lambda+2)}{\Gamma(n+\lambda+2+i)} \cdot \frac{\Gamma(n+1+i)}{\Gamma(n+1)}. \end{aligned}$$

form the sequence tending to $\lim_{r\to 1} a(r) = 1 \in S^1$, as $n \to \infty$.

Let $\lambda \to \infty$. By the asymptotic formulas for the Gamma function (see, for example, [1], p. 257) we have

$$\gamma_{a,\lambda}(n) = \frac{\Gamma(n+1+i)}{\Gamma(n+1)} (n+\lambda+2)^{-i} \left(1 + O(\frac{1}{n+\lambda})\right).$$

Thus the set $M_\infty(a)$ is the union of the circles $\mathbb{T}_n$ centered in the origin and having radiuses

$$r_n = \left|\frac{\Gamma(n+1+i)}{\Gamma(n+1)}\right| = \left|\frac{(n+i)(n-1+i)\cdots i\Gamma(i)}{n!}\right|, \qquad n \in \mathbb{Z}_+,$$

that is

$$M_\infty(a) = \bigcup_{n=0}^{\infty} \mathbb{T}_n.$$

According to formula 6.1.29 from [1] we have

$$|\Gamma(i)| = \left(\frac{\pi}{\sinh \pi}\right)^{\frac{1}{2}} = \left(\frac{2\pi}{e^\pi - e^{-\pi}}\right)^{\frac{1}{2}} \approx 0.52.$$

Thus

$$r_n = |\Gamma(i)| \prod_{j=1}^{n} \left(1 + \frac{1}{j^2}\right)^{\frac{1}{2}}.$$

That is $r_n < r_{n+1}$ for all $n \in \mathbb{Z}_+$. Moreover by the asymptotic formula for the Gamma function we have

$$\lim_{n\to\infty} r_n = 1.$$

Note that the values of a few first radiuses are as follows

$$r_0 \approx 0.52, \qquad r_1 \approx 0.73, \qquad r_2 \approx 0.82, \qquad r_3 \approx 0.86.$$

The process of forming of the limit set $M_\infty(a)$ here is somewhat different compared to the previous examples. This is caused by an oscillatory type of discontinuity of the symbol at the point $r = 0$. For each fixed and sufficiently large λ the corresponding set $\gamma_{a,\lambda}$ is a part of a spiral approaching the unit circle and ending at $a(1)$. While growing λ these spirals make longer and compress to the unit circle; for very large values of λ the points of the sequence $\gamma_{a,\lambda}(n)$ are practically collocated on the corresponding circles of the limit set, moving along these circles as λ changes. To make a more informative picture we draw the values of $\gamma_{a,\lambda}(n)$ with fixed n for different values of λ. We illustrate below the forming of the first, second and fourth circles of the limit set $M_\infty(a)$ with λ changing from 0 to 20000. Note that in the picture of the limit set the part between the last circle drawn and the unit circle is omitted.

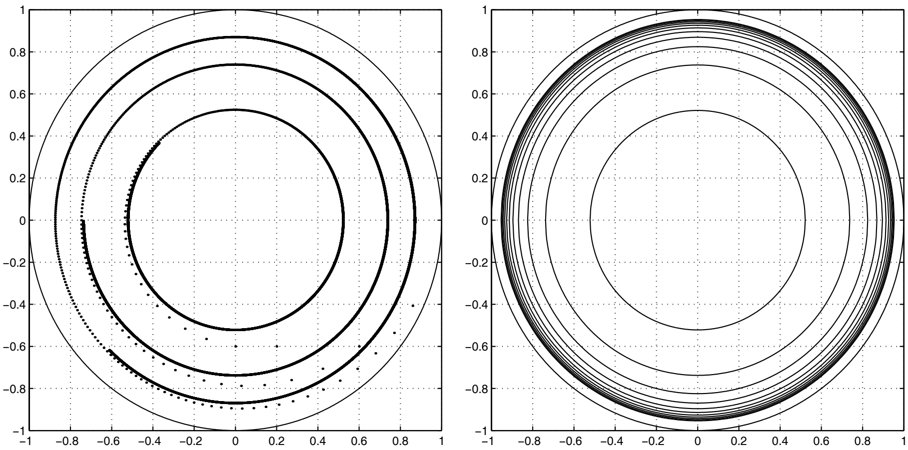


FIGURE 3. Forming of the circles $n = 0,\, 1,\, 3$ for $\lambda = 0, 1, \ldots, 20000$, and the limit set $M_\infty(a)$.

6. Spectra of Toeplitz operators, unbounded symbols

As has been already mentioned there exist unbounded radial symbols for which the corresponding Toeplitz operators are bounded on each weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$, with $\lambda \geq 0$.

Note that in spite of the fact that for each fixed λ the sequence $\{\gamma_{a,\lambda}(n)\}$ is bounded for symbols satisfying, for example, the conditions of Theorem 2.1, the estimates given in the proof are not uniform on λ. This suggests that in spite of having bounded $\operatorname{sp} T_a$ for each λ, the limit spectrum $M_\infty(a)$ may be unbounded. The next theorem shows that this observation is not accidental.

Theorem 6.1. *Let* $a(\sqrt{r}) \in L_1(0,1) \cap C[0,1)$*. Then*

$$\operatorname{Range} a \subset M_\infty(a).$$

Proof. We apply the Laplace method as in Theorem 4.1. Fix any point $r_0 \in (0,1)$, and for each $n \in \mathbb{N}$ take λ_n such that

$$\frac{\lambda_n}{n} = r_0^{-1} - 1.$$

Then by (4.4) we have

$$\gamma_{a,\lambda}(n) = a(\sqrt{r_0})(1 + \alpha(\sqrt{\lambda_n^2 + n^2})),$$

where $\lim_{L\to\infty} \alpha(L) = 0$. Thus if $n \to \infty$ then $\lambda_n \to \infty$ as well, and we have

$$a(\sqrt{r_0}) \in M_\infty(a). \qquad \square$$

Let us show now that the property (5.8), previously established for bounded symbols, remains valid for unbounded radial symbols as well.

Theorem 6.2. *Let $a(\sqrt{r}) \in L_1(0,1)$. Then*

$$M_\infty(a) \subset \mathrm{conv}(\text{ess-Range}\, a). \tag{6.1}$$

Proof. For each $M > 0$ consider the following bounded symbol

$$a_M(r) = \begin{cases} a(r), & \text{if} \quad |a(r)| \le M, \\ 0, & \text{if} \quad |a(r)| > M, \end{cases}$$

and the corresponding sequence

$$\gamma_{a_M,\lambda}(n) = \frac{1}{\mathrm{B}(n+1,\lambda+1)} \int_0^1 a_M(\sqrt{r})\,(1-r)^\lambda\, r^n dr, \qquad n \in \mathbb{Z}_+.$$

By (4.1) we have obviously

$$\{\gamma_{a_M,\lambda}(n)\} \subset \mathrm{conv}(\text{ess-Range}\, a_M) \subset \mathrm{conv}(\text{ess-Range}\, a).$$

The equality

$$\lim_{M\to\infty} \gamma_{a_M,\lambda}(n) = \gamma_{a,\lambda}(n),$$

verified by the Lebesgue dominated convergence theorem, implies that

$$\{\gamma_{a,\lambda}(n)\} \subset \mathrm{conv}(\text{ess-Range}\, a),$$

and we have (6.1). □

Note that for functions $a(\sqrt{r}) \in L_1(0,1) \cap C(0,1)$ Theorems 6.1 and 6.2 imply that

$$\mathrm{Range}\, a \subset M_\infty(a) \subset \mathrm{conv}(\mathrm{Range}\, a).$$

We now show that each of these inclusions can be an equality.

Example 6.3. For each $j \in \mathbb{N}$ define $I_j = [1 - j^{-1} - j^{-3}, 1 - j^{-1}]$, and let $\{\theta_j\}_{j\in\mathbb{N}} \subset [0, 2\pi)$ be a sequence such that $\mathrm{clos}\{\theta_j\}_{j\in\mathbb{N}} = [0, 2\pi]$. Define the symbol as follows

$$a(\sqrt{r}) = \begin{cases} je^{i\theta_j}, & r \in I_j, \quad j \in \mathbb{N} \\ 0, & r \in [0,1] \setminus \cup_{j=1}^\infty I_j \end{cases}.$$

This symbol satisfies the condition (2.4) for $j = 1$, and thus the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded for every $\lambda \ge 0$. Indeed

$$\begin{aligned} |B^{(1)}(r)| &= \left| \int_r^1 a(\sqrt{s}) ds \right| \le \int_r^1 |a(\sqrt{s})|\, ds \le \sum_{1-j^{-1}>r} j \cdot j^{-3} \\ &= \sum_{j>(1-r)^{-1}} j^{-2} \le \mathrm{const}\,(1-r). \end{aligned}$$

Further, $a(\sqrt{r}) = je^{i\theta_j}$ for $r \in I_j$, and $a(\sqrt{r}) = 0$ for $r \in (1 - j^{-1}, 1 - (j+1)^{-1} - (j+1)^{-3}]$, which implies just in the same way as in Theorem 5.1 that the straight line segment $[0, je^{i\theta_j}]$ belongs to $M_\infty(a)$.

Thus

$$\mathrm{Range}\, a \subset M_\infty(a) = \mathbb{C} = \mathrm{conv}(\mathrm{Range}\, a).$$

Example 6.4. Given $\alpha \in (0,1)$ introduce $a(\sqrt{r}) = r^{i-\alpha}$. Calculate

$$\begin{aligned} \gamma_{a,\lambda}(n) &= \frac{1}{\mathrm{B}(n+1,\lambda+1)} \int_0^1 (1-r)^\lambda \, r^{n+i-\alpha} dr \\ &= \frac{\mathrm{B}(n+1+i-\alpha,\lambda+1)}{\mathrm{B}(n+1,\lambda+1)} = \frac{\Gamma(n+\lambda+2)}{\Gamma(n+\lambda+2+i-\alpha)} \cdot \frac{\Gamma(n+1+i-\alpha)}{\Gamma(n+1)}, \end{aligned}$$

and it is easy to see that for each fixed λ the sequence $\{\gamma_{a,\lambda}(n)\}$ is bounded, which implies the boundedness of the corresponding Toeplitz operator.

Let now $\lambda \to \infty$. Using the asymptotic formulas for the Gamma function (see, for example, [1], p. 257) we have

$$\gamma_{a,\lambda}(n) = \frac{\Gamma(n+1+i-\alpha)}{\Gamma(n+1)} \, (n+\lambda+2)^{\alpha-i} \left(1 + O(\frac{1}{n+\lambda})\right).$$

If n is fixed, or bounded this expression gives as only one point from the set $M_\infty(a)$, namely ∞. Let now both n and λ tend to infinity. Then we have

$$\begin{aligned} \gamma_{a,\lambda}(n) &= \left(\frac{n+\lambda+2}{n+1}\right)^{\alpha-i} \left(1+O(\frac{1}{n+1})\right) \left(1+O(\frac{1}{n+\lambda})\right) \\ &= \left(1+\frac{\lambda+1}{n+1}\right)^{\alpha-i} \left(1+O(\frac{1}{n+1}) + O(\frac{1}{n+\lambda})\right). \end{aligned}$$

Given arbitrary $u \in (0,1)$, take λ and n in such a form that

$$\left(1+\frac{\lambda+1}{n+1}\right) = u^{-1}.$$

Thus

$$\gamma_{a,\lambda}(n) = u^{i-\alpha} \left(1+O(\frac{1}{n+1}) + O(\frac{1}{n+\lambda})\right),$$

and thus in this case

$$\mathrm{Range}\, a = M_\infty(a) \subset \mathrm{conv}(\mathrm{Range}\, a).$$

Let us mention some peculiarities of this example. In spite of the *unboundedness* of the symbol, the Toeplitz operator $T_a^{(\lambda)}$ is *bounded* for each λ, and the sequence $\gamma_{a,\lambda}$ is *bounded* by $\|T_a^{(\lambda)}\|$ for each λ. The rough estimate here is

$$\sup_n |\gamma_{a,\lambda}(n)| = \|T_a^{(\lambda)}\| \approx \mathrm{const}\, \lambda^\alpha.$$

That is, being bounded for each λ the norm and $\sup_n |\gamma_{a,\lambda}(n)|$ grow as λ tends to infinity. The tendency of $\gamma_{a,\lambda}$ to approach $M_\infty(a)$ here is developed first of all in the growth of the size (i.e., $\sup_n |\gamma_{a,\lambda}(n)|$) of $\mathrm{sp}\, T_a^{(\lambda)}$ while λ tends to infinity.

In the next figure setting $\alpha = 0.1$ we present the sequence $\gamma_{a,\lambda}$ for $\lambda = 100000$ and the limit set $M_\infty(a) = \mathrm{Range}\, a$.

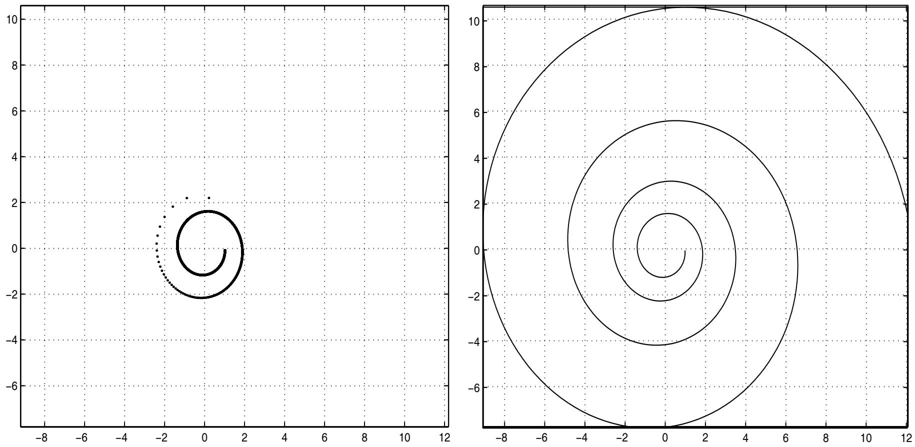


FIGURE 4. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ for $\lambda = 100000$ and the limit set $M_\infty(a)$.

References

[1] Milton Abramowith and Irene A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables.* New York : Dover Publications, 1965.

[2] F. A. Berezin. Quantization, *Math. USSR Izvestija*, **8** (1974), 1109–1165.

[3] F. A. Berezin. General Concept of Quantization, *Commun. Math. Phys.*, **40** (1975), 153–174.

[4] F. A. Berezin. *Method of Second Quantization.* “Nauka”, Moscow, 1988.

[5] M .V. Fedoriuk. *Saddle-Point Mathod.* Nauka, Moskow, 1977.

[6] I. Gohberg and N. Krupnik. On the algebra generated by one–dimensional singular integral operators with piece–wise continuous coefficients, *Funct. Analysis Appl.*, **4** (1970), 193–201.

[7] S. Grudsky, A. Karapetyants and N. Vasilevski. Toeplits operators on the unit ball in $\mathbb{C}^n$ with radial symbols. Reporte Interno #292, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 2001. (to appear in *J. Oper. Th.*)

[8] S. Grudsky and N. Vasilevski. Bergman-Toeplitz operators: Radial component influence. *Integr. Equat. Oper. Th.*, **40** (2001), 16–33.

[9] R. Hagen, S. Roch and B. Silbermann. *C^*-Algebras and Numerical Analysis.* Marcel Dekker, Inc., New York, Basel, 2001.

[10] H. Hedenmalm, B. Korenblum and K. Zhu. *Theory of Bergman Spaces.* Springer Verlag, New York, Inc., 2000.

[11] S. Klimek and A. Lesniewski. Quantum Rieman Surfaces I. The Unit Disk, *Commun. Math. Phys.*, **146** (1992), 103–122.

[12] B. Korenblum and K. Zhu. An application of Tauberian theorems to Toeplitz operators. *J. Operator Theory*, **39** (1995), 353–361.

[13] Jie Miao. Toeplitz operators with bounded radial symbols on the harmonic Bergman space of the unit ball. *Acta Sci. Math. (Szeged)*, **63** (1997), 639–645.

[14] Sangadji and K. Stroethoff. Compact Toeplitz operators on generalized Fock spaces. *Acta Sci. Math. (Szeged)*, **64** (1998), 657–669.

[15] K. Stroethoff. Compact Toeplitz operators on Bergman spaces. *Math. Poc. Camb. Phil. Soc.*, **124** (1998), 151–160.

[16] K. Stroethoff. Compact Toeplitz operators on weighted harmonic Bergman spaces. *J. Austal. Math. Soc. (Series A)*, **64** (1998), 136–148.

[17] N. L. Vasilevski. Banach algebras generated by two-dimensional integral operators with Bergman kernel and piece-wise continuous coefficients. I, *Soviet Math. (Izv. VUZ)*, **30** (1986), 14–24.

[18] N. L. Vasilevski. Bergman Space Structure, Commutative Algebras of Toeplitz Operators and Hyperbolic Geometry. Reporte Interno #284, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 2000. (to appear in *Integr. Equat. Oper. Th.*)

[19] N. L. Vasilevski. Toeplitz Operators on the Bergman Spaces: Inside-the-Domain Effects. *Contemp. Math.*, **289** (2001), 79–146.

[20] Kehe Zhu. Positive Toeplitz operators on weighted Bergman space. *J. Oper. Th.*, **20** (1988), 329–357.

S. Grudsky
Departamento de Matemáticas
CINVESTAV del I.P.N.
Apartado Postal 14-740
07000 México, D.F.
México
e-mail: `grudsky@math.cinvestav.mx`

A. Karapetyants
Department of Mathematics
Rostov-on-Don State University
Rostov-on-Don
Russia
e-mail: `tiskidar@aaanet.ru`

N. Vasilevski
Departamento de Matemáticas
CINVESTAV del I.P.N.
Apartado Postal 14-740
07000 México, D.F.
México
e-mail: `nvasilev@math.cinvestav.mx`

Submitted: September 15, 2003