PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 6, June 2014, Pages 2079–2086
S 0002-9939(2014)11943-5
Article electronically published on March 10, 2014

ON POSITIVE TYPE INITIAL PROFILES FOR THE KDV EQUATION

SERGEI GRUDSKY AND ALEXEI RYBKIN

(Communicated by James E. Colliander)

ABSTRACT. We show that the KdV flow evolves any real locally integrable initial profile q of the form $q = r' + r^2$, where $r \in L^2_{\text{loc}}$, $r|_{\mathbb{R}_+} = 0$ into a meromorphic function with no real poles.

1. INTRODUCTION

This note is closely related to the recent paper [7] by Kappeler et al. and [12] by one of the authors.

More specifically, we are concerned with well-posedness and related issues of the initial value problem for the Korteweg-De Vries (KdV) equation ($x \in \mathbb{R}, t \geq 0$)

$$\begin{cases} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \\ u(x,0) = q(x) \end{cases} \tag{1.1}$$

with certain non-smooth and non-decaying initial profiles q.

The problem of well-posedness of (1.1) was raised back in the late 1960s about the same time as the inverse scattering formalism for (1.1) was discovered and has drawn enormous attention. We are not in a position to go over the extensive literature on the subject and refer to the book [13] by Tao where further references are given. We only mention here that a large amount of effort has been put into well-posedness in Sobolev spaces $H^s(\mathbb{R})$ with negative indices s (that is, the space of distributions subject to $(1+|x|)^s \widehat{f}(x) \in L^2(\mathbb{R})$). Reaching $s = -3/4$ in [3] by Colliander et al. was a very important milestone. This case covers such physically significant initial data as the delta function, Coulomb potential, etc., but the harmonic analytical methods employed there meet real problems for $s < -3/4$. An important step in crossing this threshold was done by Kappeler et al. [7], where it was shown that (1.1) is globally well-posed in a certain sense if $q = r' + r^2$ with some $r \in L^2(\mathbb{R})$.

The map

$$B(r) = r' + r^2, \quad r \in L^2_{\text{loc}}(\mathbb{R}),$$

is commonly referred to as Miura. It is easy to see that the Schrödinger operator

$$L_q = -\partial_x^2 + q(x), \quad x \in \mathbb{R},$$

Received by the editors July 10, 2012.

2010 *Mathematics Subject Classification.* Primary 34B20, 37K15, 47B35.

Key words and phrases. KdV equation, Titchmarsh-Weyl m-function, Hankel operators, Miura map.

The first author was partially supported by PROMEP (México) via "Proyecto de Redes" and by CONACYT grant 102800.

The second author was supported in part by the NSF under grant DMS 1009673.

is positive, i.e. $L_q \geq 0$, for any such $q \in B\left(L^2_{\text{loc}}(\mathbb{R})\right)$. One of the main results of [7] is that the converse of this fact also holds true; i.e. if $L_q \geq 0$, then $q \in B\left(L^2_{\text{loc}}(\mathbb{R})\right)$. We call such a potential q positive type, and it is our main object.

We note that all functions in $B\left(L^2(\mathbb{R})\right)$ exhibit certain decay at $\pm\infty$. On the other hand, there has been significant interest in non-decaying solutions to (1.1) (other than periodic). The case of the so-called steplike initial profiles (i.e. when $q(x) \to 0$ sufficiently fast as $x \to +\infty$ $(-\infty)$ and $q(x)$ doesn't decay at $-\infty$ $(+\infty)$) is of physical interest and has attracted much attention since the early 1970s. We refer to the recent paper [5] by Egorova-Grunert-Teschl for a comprehensive account of the (rigorous) literature on steplike initial profiles with specified behavior at infinity (e.g. q tending to a constant, periodic function, etc.). In the recent paper [12] of one of the authors (see also [11]) the case of q's rapidly decaying at $+\infty$ and sufficiently arbitrary at $-\infty$ is studied in great detail.

The current note is concerned with treating positive type steplike initial data q in (1.1). To avoid technical complications we assume that $q \in L^1_{\text{loc}}(\mathbb{R})$ and identically vanishes on $\mathbb{R}_+ := (0, \infty)$, even though q has no smoothness and any kind of decay at $-\infty$ except that the fact that q vanishes on $\mathbb{R}_+$ leads to an extremely strong smoothing effect. Dispersion instantaneously turns such initial profiles $q(x)$ into a function $u(x,t)$ meromorphic in x on the whole complex plane for any $t > 0$. The well-posedness of problem (1.1) can therefore be understood in a classical sense and moreover it comes with an explicit formula

$$u(x,t) = -2\partial_x^2 \log\det\left(1 + \mathbb{H}_{x,t}\right),$$

where $\mathbb{H}_{x,t}$ is the Hankel operator with symbol

$$\varphi_{x,t}(\lambda) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)} e^{2i\lambda(4\lambda^2 t + x)}, \quad \lambda \in \mathbb{R}\ ,\ x \in \mathbb{R}\ ,\ t > 0,$$

where m is the Titchmarsh-Weyl m-function associated with L_q on $\mathbb{R}_- := (-\infty, 0)$ with a Dirichlet boundary condition at 0.

Each pole of the meromorphic in x function $u(x,t)$ is necessarily double moving over time $t > 0$, but no other type of singularity can develop. The well-posedness of our problem, among others, means that such poles never cross the real line (but may however asymptotically approach it). That is, no positon (a solution having a second order moving singularity) may occur in our situation. Incidentally, a positon solution is an example of a singular solution from $H^{-2}(\mathbb{R})$. Note that while such solutions (as well as some other strongly singular solutions) were discovered a long time ago and have received considerable attention (see e.g. [9] by Matveev and the extensive literature cited therein), no rigorous general theory is available to date to treat such initial data. We only mention that the main difficulty comes from the fact that the Schrödinger operator L_q with $q \in H^{-2}$ is ill-defined, as it can be introduced in different non-equivalent ways which result in real problems applying the inverse scattering method. Standard harmonic analytic techniques also fail to handle such singular distributions.

Our approach is based on a suitable adaptation of the inverse scattering formalism and analysis of Hankel operators with certain oscillatory symbols. Moreover, the theory of Hankel operators is used in a quite substantial way, and we hope that our approach may turn out to be productive in many other important issues of completely integrable systems.

The paper is organized as follows. In Section 2 we review Hankel operators and prove a new result related to a Hankel operator with a cubic oscillatory symbol. In Section 3 we recall the concept of the Titchmarsh-Weyl m-function and some elements of the scattering theory. In the last section, Section 4, we state and prove our main result.

2. Hankel operators

Hankel operators naturally appear in linear algebra, operator theory, complex analysis, mathematical physics, and many other areas. In our note they play a crucial role. However, their formal definitions vary. In the context of integral operators, a Hankel operator is usually defined as an integral operator on $L^2(\mathbb{R}_+)$ whose kernel depends on the sum of the arguments; i.e.

$$(2.1)\qquad (\mathbb{H}f)(x) = \int_0^\infty H(x+y)f(y)dy \quad , \quad x \geq 0\ ,\ f \in L^2(\mathbb{R}_+)\ ,$$

with some function H.

In many situations, including ours, H is not a function but rather a distribution. It is convenient then to accept a regularized version of (2.1).

Let[1]

$$(\mathcal{F}f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int e^{i\lambda x} f(x)dx$$

be the Fourier transform and χ the Heaviside function of $\mathbb{R}_+$.

Definition 2.1. Given $\varphi \in L^\infty(\mathbb{R})$, we call the operator $\mathbb{H}_\varphi$ on $L^2(\mathbb{R}_+)$ defined for any $f \in L^2(\mathbb{R}_+)$ by

$$(2.2)\qquad \mathbb{H}_\varphi f = \chi \mathcal{F} \varphi \mathcal{F} f$$

the Hankel operator on $L^2(\mathbb{R}_+)$ with symbol φ.

It follows from a straightforward computation that (2.1) and (2.2) agree if $\varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $H = \mathcal{F}\varphi$. However if φ is merely $L^\infty(\mathbb{R})$, then $\mathcal{F}\varphi$ is not a function but a (tempered) distribution. The operator $\mathbb{H}$ given by (2.1) is no longer well-defined, but the one given by (2.2) is.

The Hankel operator $\mathbb{H}_\varphi$ is clearly bounded from (2.2). One immediately has

$$(2.3)\qquad \|\mathbb{H}_\varphi\| \leq \|\varphi\|_\infty\,.$$

Membership of $\mathbb{H}_\varphi$ in narrower Schatten-Von Neumann ideals is, however, a much more subtle issue which was completely resolved by Peller in about 1980 (see e.g. [10]).

We will be particularly concerned with the invertibility of $1 + \mathbb{H}_\varphi$. The first fact is trivial.

Lemma 2.2. *Let φ be such that $|\varphi(\lambda)| \leq 1$ a.e. $\lambda \in \mathbb{R}$ and $|\varphi(\lambda)| < 1$ a.e. on a set S of positive Lebesgue measure. Then -1 is not an eigenvalue of $\mathbb{H}_\varphi$.*

Proof. Assume -1 is an eigenvalue of $\mathbb{H}_\varphi$ and $f \neq 0$ is the corresponding normalized eigenvector (i.e. $\|f\|_{L^2(\mathbb{R}_+)} = 1$).

It follows from

$$f + \mathbb{H}_\varphi f = 0$$

[1]For brevity we set $\int := \int_{-\infty}^{\infty}$.

that

$$1+\int \varphi(\lambda)\widehat{f}(\lambda)\widehat{\overline{f}}(\lambda)d\lambda = 0$$

and hence

$$1+\operatorname{Re}\int \varphi(\lambda)\widehat{f}(\lambda)\widehat{\overline{f}}(\lambda)d\lambda = 0. \tag{2.4}$$

But

$$\begin{aligned}\operatorname{Re}\int \varphi(\lambda)\widehat{f}(\lambda)d\lambda &\le \int \left|\varphi(\lambda)\widehat{f}(\lambda)\right|\cdot\left|\widehat{\overline{f}}(\lambda)\right|d\lambda \\ &\le \left\|\varphi\widehat{f}\right\|_{L^2(\mathbb{R}_+)}\cdot\left\|\widehat{\overline{f}}\right\|_{L^2(\mathbb{R}_+)} = \left\|\varphi\widehat{f}\right\|_{L^2(\mathbb{R}_+)} \\ &= \left(1-\int_S\left(1-|\varphi(\lambda)|^2\right)\left|\widehat{f}(\lambda)\right|^2 d\lambda\right)^{1/2} \\ &< 1, \qquad (2.5)\end{aligned}$$

since as an H^2 function $\widehat{f}$ cannot vanish identically on S. Comparing (2.4) and (2.5) leads to a contradiction. □

The proof of Lemma 2.2 is no longer valid if $|\varphi(\lambda)| = 1$ for a.e. real λ.

However in our setting symbols φ have a very specific structure:

$$\varphi(\lambda) = e^{i\lambda(\lambda^2+a)}I(\lambda), \tag{2.6}$$

where a is a real number and I is an inner function of the upper half plane (i.e.[2] $I \in H^\infty_+$ and $|I(\lambda)| = 1$ a.e. $\lambda\in\mathbb{R}$).

Lemma 2.3. *Let φ be given by* (2.6). *Then*

(1) $\mathbb{H}_\varphi$ *is a compact operator,*

(2) $1+\mathbb{H}_\varphi$ *is invertible.*

Proof. Our argument is based upon the factorization (see [1], [4], Section 5.10, [6])

$$e^{i\lambda(\lambda^2+a)} = B(\lambda)U(\lambda), \quad \lambda\in\mathbb{R}, \tag{2.7}$$

where $B(\lambda)$ is a Blaschke product with infinitely many zeros accumulating at infinity and U is a unimodular function from $C(\overline{\mathbb{R}})$, the class of continuous on $\mathbb{R}$ functions f subject to

$$\lim_{\lambda\to-\infty} f(\lambda) = \lim_{\lambda\to\infty} f(\lambda) \ne \pm\infty.$$

Since a product of an inner function and a $C(\overline{\mathbb{R}})$-function is in the algebra $H^\infty_+ + C(\overline{\mathbb{R}})$, by the Hartman theorem [8] $\mathbb{H}_\varphi$ is compact and (1) is proven.

Consider the Hankel operator (2.2) in the Fourier representation. Denoting $P_\pm$ as the Riesz projection in $L^2(\mathbb{R}_+)$ onto $H^2_\pm$, we have

$$\begin{aligned}\mathcal{F}\mathbb{H}_\varphi\mathcal{F}^{-1} &= \mathcal{F}\chi\mathcal{F}\varphi\mathcal{F}\mathcal{F}^{-1} \\ &= P_+\mathcal{F}\mathcal{F}\varphi = P_+\mathcal{F}^2\varphi \\ &= P_+J\varphi = JP_-\varphi,\end{aligned}$$

where $Jf(x) = f(-x)$. Thus, the operator

$$JP_-\varphi : H^2_+ \to H^2_+ \tag{2.8}$$

[2] $H^p_\pm$ $(0 < p \le \infty)$ are standard Hardy spaces of the upper (lower) half planes $\mathbb{C}^\pm$.

is unitarily equivalent to $\mathbb{H}_\varphi$. Let $\mathbb{T}_\varphi$ be the Toeplitz operator on H^2_+, i.e.

$$\mathbb{T}_\varphi f = P_+ \varphi f, \quad f \in H^2_+.$$

Note ([2], Ch. 2) that (2.7) implies left-invertibility of the operator $\mathbb{T}_\varphi$ and, by the Devinatz-Widom theorem ([2], p. 59), there exists a function $f \in H^\infty_+$ such that

$$\|\varphi - f\|_{L^\infty} < 1.$$

Thus, it immediately follows from the representation (2.8) that

$$\mathbb{H}_\varphi = \mathbb{H}_{\varphi - f}$$

and hence from (2.3) and (2.7) that

$$\|H_\varphi\| = \|\mathbb{H}_{\varphi - f}\| \leq \|\varphi - f\|_{L^\infty} < 1.$$

This proves (2), and the lemma is proven. □

3. The Titchmarsh-Weyl m-function and the reflection coefficient

Consider the half line Schrödinger equation

$$L_q u = zu, \ x \in \mathbb{R}_-.$$

If $L_q \geq 0$, then there exists a unique (up to a multiplicative constant) solution, called Weyl, such that $\Psi(x, z) \in L^2(\mathbb{R}_-)$ for each $z \in \mathbb{C}^+$. Existence and uniqueness of such a solution take place under much more general assumptions on q (see e.g. [14]), but we don't need to explain them here.

Definition 3.1. The function

$$m(z) = -\frac{\partial_x \Psi(0, z)}{\Psi(0, z)}, \quad z \in \mathbb{C}^+,$$

is called a (Dirichlet or principal) Titchmarsh-Weyl m-function associated with L_q on $L^2(\mathbb{R}_-)$ with the Dirichlet boundary condition $u(0) = 0$.

The Titchmarsh-Weyl m-function is a fundamental object of the spectral theory of ordinary differential operators and has a number of important properties. In particular, $m(z)$ is analytic on $\mathbb{C}^+$ and has the Herglotz property: $m : \mathbb{C}^+ \to \mathbb{C}^+$. Moreover, due to the positivity of L_q the function $m(\lambda^2)$ as a function of λ is analytic everywhere in $\mathbb{C}^+$ and $m(\lambda^2) = i\lambda + o(1), \lambda \to \infty$, in any angle $0 < \varepsilon < \arg \lambda < \pi - \varepsilon$.

Now define the reflection coefficient $R(\lambda)$ from the right incident of q such that $q|_{\mathbb{R}_+} = 0$. To this end consider a solution to

$$L_q y = \lambda^2 y$$

which is proportional to the Weyl solution $\Psi(x, \lambda^2)$ on $\mathbb{R}_-$ and equal to $e^{-i\lambda x} + R(\lambda) e^{i\lambda x}$ on $\mathbb{R}_+$. From the continuity of this solution and its derivative at 0 one has

$$R(\lambda) = \frac{i\lambda - \frac{\psi'(0,\lambda^2)}{\psi(0,\lambda^2)}}{i\lambda + \frac{\psi'(0,\lambda^2)}{\psi(0,\lambda^2)}} = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)}. \tag{3.1}$$

Note that while m has no clear physical meaning, the reflection coefficient R does.

It follows from the above properties of the m-function that $R(\lambda)$ is an analytic function on $\mathbb{C}^+$ and $|R(\lambda)| \leq 1$ for $\lambda \in \mathbb{C}^+$.

4. Main result

In this section we state and prove our main result.

Theorem 4.1. *Let q in* (1.1) *be locally integrable, supported on $\mathbb{R}_-$ and such that the Schrödinger operator $L_q \geq 0$. Then there is a (unique) classical solution to* (1.1) *given by*

$$u(x,t) = -2\partial_x^2 \log\det\left(1+\mathbb{H}_{x,t}\right). \tag{4.1}$$

Here $\mathbb{H}_{x,t}$ is the trace class Hankel operator on $L^2(\mathbb{R}_+)$ with the symbol

$$\varphi_{x,t}(\lambda) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)} e^{2i\lambda x + 8i\lambda^3 t},$$

where m is the (Dirichlet) Titchmarsh-Weyl m-function of L_q on $L^2(\mathbb{R}_-)$.

The solution $u(x,t)$ is meromorphic in $\mathbb{C}^+$ for any $t>0$ with no real poles.

Proof. It is proven in [7] that

$$L_q \geq 0 \quad \Rightarrow \quad q \in B\left(L^2_{\text{loc}}\right),$$

where $B(r) = r' + r^2$ is the Miura map. Since $q \in L^1_{\text{lim func}\,loc}$, the function r is locally absolutely continuous. Approximate r with smooth, compactly supported functions $\tilde{r}$. Then $\tilde{q} = \tilde{r}' + \tilde{r}^2$ approximates q in L^1_{loc}. For each $\tilde{q}$ there exists the right reflection coefficient $\widetilde{R}$. The (classical) Marchenko operator $\widetilde{\mathbb{H}}_{x,t}$ has no discrete component (since $L_{\tilde{q}} \geq 0$) and hence it takes the form

$$\left(\widetilde{\mathbb{H}}_{x,t} f\right)(\cdot) = \int_0^\infty \widetilde{H}_{x,t}(\cdot + y) f(y)dy, \tag{4.2}$$

where

$$\widetilde{H}_{x,t}(\cdot) = \frac{1}{2\pi}\int e^{2i\lambda x + 8i\lambda^3 t} e^{i\lambda(\cdot)} \widetilde{R}(\lambda)d\lambda. \tag{4.3}$$

For the reflection coefficient $\widetilde{R}$, by (3.1), we have

$$\widetilde{R}(\lambda) = \frac{i\lambda - \widetilde{m}(\lambda^2)}{i\lambda + \widetilde{m}(\lambda^2)},$$

where $\tilde{m}$ is the Titchmarsh-Weyl m-function associated with $L_{\tilde{q}}$ with the Dirichlet boundary condition at 0. By the properties of the reflection coefficient, $\widetilde{R}(\lambda)$ is analytic in $\mathbb{C}^+$, $\widetilde{R}(\lambda) = O(1/\lambda)$, $\lambda \to \pm\infty$, and $\left|\widetilde{R}(\lambda)\right| \leq 1$, $\lambda \in \mathbb{C}^+$. One can obviously deform the contour of integration in (4.3) now and thus (4.3) reads

$$\widetilde{H}_{x,t}(\cdot) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = h} e^{2i\lambda x + 8i\lambda^3 t} e^{i\lambda(\cdot)} \widetilde{R}(\lambda)d\lambda, \tag{4.4}$$

for any $h>0$. Since the integrand in (4.4) is clearly integrable along the line $\operatorname{Im}\lambda = h$, the operator $\widetilde{\mathbb{H}}_{x,t}$ is trace class (see [12]) and the function

$$\tilde{u}(x,t) = -2\partial_x^2 \log\det\left(1+\widetilde{\mathbb{H}}_{x,t}\right) \tag{4.5}$$

is well-defined and solves (1.1) with the initial data $\tilde{q}$.

We now pass to the limit in (4.5) as $\tilde{q} \to q$ in L^1_{loc}. It is well-known that $\widetilde{m}(\lambda^2) \to m(\lambda^2)$ on each compact set in $\mathbb{C}^+$, and hence

$$\widetilde{R}(\lambda) = \frac{i\lambda - \widetilde{m}(\lambda^2)}{i\lambda + \widetilde{m}(\lambda^2)} \quad \longrightarrow \quad R(\lambda) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)}$$

on each compact in $\mathbb{C}^+$. The oscillatory factor $e^{2i\lambda x+8i\lambda^3 t}$ exhibits a superexponential decay on $\operatorname{Im}\lambda = h > 0$. This means that (see [12] for)

$$\widetilde{\mathbb{H}}_{x,t} \longrightarrow \mathbb{H}_{x,t}$$

for any $x \in \mathbb{R}$, $t > 0$ in trace class norm, and hence

$$\det\left(1+\widetilde{\mathbb{H}}_{x,t}\right) \longrightarrow \det\left(1+\mathbb{H}_{x,t}\right).$$

Note that $\widetilde{H}_{x,t}$ and

$$H_{x,t}(\cdot) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=h} e^{2i\lambda x+8i\lambda^3 t}e^{i\lambda(\cdot)}R(\lambda)d\lambda$$

are clearly entire with respect to x, for any $t > 0$. It is quite easy to see that $\widetilde{\mathbb{H}}_{x,t}, \mathbb{H}_{x,t}$ are operator-valued functions entire with respect to x for any $t > 0$. This means that the functions

$$\tilde{u}(x,t) = -2\partial_x^2 \log\det\left(1+\widetilde{\mathbb{H}}_{x,t}\right)$$

are meromorphic in x on the whole complex plane for any $t > 0$ and converge to the meromorphic function

$$u(x,t) = -2\partial_x^2 \log\det\left(1+\mathbb{H}_{x,t}\right)$$

as $\tilde{q} \to q$ in L^1_{loc}.

It remains to show that $\det(1+\mathbb{H}_{x,t})$ doesn't vanish on the real line for any $t > 0$. Since $\mathbb{H}_{x,t}$ is trace class, this amounts to showing that -1 is not an eigenvalue of $\mathbb{H}_{x,t}$ for all $x \in \mathbb{R}$, $t > 0$. We have two cases: L_q has some absolutely continuous (a.c.) spectrum and L_q has no a.c. spectrum. The first case immediately follows from Lemma 2.2.

The second case is a bit more involved. If the a.c. spectrum of L_q is empty, then the Titchmarsh-Weyl m-function is real a.e. on the real line and hence the reflection coefficient $|R(\lambda)| \leq 1$ in $\mathbb{C}^+$ and $|R(\lambda)| = 1$ a.e. on $\mathbb{R}$; i.e. R is an inner function of the upper half plane. Lemma 2.3 then applies. □

Remark 4.2. Theorem 4.1 implies very strong well-posedness of the KdV equation with eventually any steplike positive type initial data supported on $(-\infty, 0)$. Each such solution $u(x,t)$ is smooth and hence solves the KdV equation in the classical sense. It also has a continuity property in the sense that if $\{q_n\}$ is a sequence of smooth functions of the form $q_n = r_n^2 + r_n'$ convergent in L^1_{loc} to q, then the sequence of the corresponding solutions $\{u_n(x,t)\}$ converges to $u(x,t)$ in L^1_{loc} for any $t > 0$. This, in turn, implies uniqueness. The initial condition is satisfied in the sense that

$$\|u(\cdot,t) - q\|_{L^1_{\text{loc}}} \to 0 \quad , \quad t \to 0.$$

Remark 4.3. We assumed that $q \in L^1_{\text{loc}}$ and $q|_{\mathbb{R}_+} = 0$ for simplicity. These conditions can be replaced with $L_q \geq 0$ and a suitable decay assumption at $+\infty$, but the consideration becomes much more involved due to serious technical complications related to singular potentials. We plan to return to this topic elsewhere.

References

[1] A. Böttcher, S. Grudsky, and I. Spitkovsky, *Toeplitz operators with frequency modulated semi-almost periodic symbols*, J. Fourier Anal. Appl. **7** (2001), no. 5, 523–535, DOI 10.1007/BF02511224. MR1845102 (2002e:47029)

[2] Albrecht Böttcher and Bernd Silbermann, *Analysis of Toeplitz operators*, 2nd ed., prepared jointly with Alexei Karlovich, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006. MR2223704 (2007k:47001)

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$*, J. Amer. Math. Soc. **16** (2003), no. 3, 705–749 (electronic), DOI 10.1090/S0894-0347-03-00421-1. MR1969209 (2004c:35352)

[4] Vladimir Dybin and Sergei M. Grudsky, *Introduction to the theory of Toeplitz operators with infinite index*, translated from the Russian by Andrei Iacob, Operator Theory: Advances and Applications, vol. 137, Birkhäuser Verlag, Basel, 2002. MR1942445 (2003j:47031)

[5] Iryna Egorova, Katrin Grunert, and Gerald Teschl, *On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data. I. Schwartz-type perturbations*, Nonlinearity **22** (2009), no. 6, 1431–1457, DOI 10.1088/0951-7715/22/6/009. MR2507328 (2010j:35447)

[6] Sergei M. Grudsky, *Toeplitz operators and the modelling of oscillating discontinuities with the help of Blaschke products*, Problems and methods in mathematical physics (Chemnitz, 1999), Oper. Theory Adv. Appl., vol. 121, Birkhäuser, Basel, 2001, pp. 162–193. MR1847211 (2002e:47033)

[7] Thomas Kappeler, Peter Perry, Mikhail Shubin, and Peter Topalov, *The Miura map on the line*, Int. Math. Res. Not. **50** (2005), 3091–3133, DOI 10.1155/IMRN.2005.3091. MR2189502 (2006k:37191)

[8] Nikolai K. Nikolski, *Operators, functions, and systems: an easy reading, Vol. 1. Hardy, Hankel, and Toeplitz*, translated from the French by Andreas Hartmann, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002. MR1864396 (2003i:47001a)

[9] V. B. Matveev, *Positons: slowly decreasing analogues of solitons* (Russian, with Russian summary), Teoret. Mat. Fiz. **131** (2002), no. 1, 44–61, DOI 10.1023/A:1015149618529; English transl., Theoret. and Math. Phys. **131** (2002), no. 1, 483–497. MR1931054 (2003h:37132)

[10] Vladimir V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. MR1949210 (2004e:47040)

[11] Alexei Rybkin, *Meromorphic solutions to the KdV equation with non-decaying initial data supported on a left half line*, Nonlinearity **23** (2010), no. 5, 1143–1167, DOI 10.1088/0951-7715/23/5/007. MR2630095 (2011i:35223)

[12] Alexei Rybkin, *The Hirota τ-function and well-posedness of the KdV equation with an arbitrary step-like initial profile decaying on the right half line*, Nonlinearity **24** (2011), no. 10, 2953–2990, DOI 10.1088/0951-7715/24/10/015. MR2842104 (2012i:37118)

[13] Terence Tao, *Nonlinear dispersive equations*, Local and global analysis, CBMS Regional Conference Series in Mathematics, vol. 106, published for the Conference Board of the Mathematical Sciences, Washington, DC, by the Amer. Math. Soc., Providence, RI, 2006. MR2233925 (2008i:35211)

[14] Gerald Teschl, *Mathematical methods in quantum mechanics. With applications to Schrödinger operators*, Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009. MR2499016 (2010h:81002)

Departamento de Matematicas, CINVESTAV del I.P.N. Aportado Postal 14-740, 07000 Mexico, D.F., Mexico

E-mail address: `grudsky@math.cinvestav.mx`

Department of Mathematics and Statistics, University of Alaska Fairbanks, P.O. Box 756660, Fairbanks, Alaska 99775

E-mail address: `arybkin@alaska.edu`