Admissions Examination for Master's / Doctorate
21 November 2018

Name: ____

Instructions: For each question encircle the correct answers. There may be several correct solutions for a single question (choose all, but points will be taken off for incorrectly chosen options). You may make calculations on the paper you are given, but you do not need to hand that in. The examination has 30 questions. We suggest reading all of the statements first. **Calculators or cell phones not allowed.**

Duration of examination: 2 hours

1. Which of the following sets is a vector space?
   (a) The set of positive real numbers with the sum given by $x \oplus y = xy$ and the multiplication by scalars given by $a \otimes x = x^a$.
   (b) The set of 2×2 matrices A such that $\det(A) = 0$;
   (c) The set of polinomials $p(x)$ with $\int_{-1}^{1} p(x)dx = 0$;
   (d) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 3x_3 = 2\}$;

2. A vector space V has 4 vectors which generate it but are linearly dependent. From this information it follows that:
   (a) $\dim(V) = 3$;
   (b) $\dim(V) \leq 3$;
   (c) $\dim(V) = 4$;
   (d) $\dim(V) < 3$.

3. Let A be a 3×3 matrix with $\det(A) = 0$. Which of the following statements are true?
   (a) $Ax = 0$ has a nontrivial solution.
   (b) $Ax = b$ has a solution for all b.
   (c) For every 3×3 matrix B, $\det(A + B) = \det(A) + \det(B)$.
   (d) For every 3×3 matrix B, $\det(AB) = 0$.

4. Which of the following are vector subspaces?

   (a) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property that $2x_1 - x_2 = 0$ and $3x_3 - x_4 = 0$.

   (b) The set of all vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$ with the property that $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

   (c) The set of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with the property that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$.

   (d) The set of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with the property that $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 0$.

5. Calculate the determinant of $\begin{pmatrix} x^1 & x^2 & x^3 \\ x^8 & x^9 & x^4 \\ x^7 & x^6 & x^5 \end{pmatrix}$.

   (a) 0;

   (b) $-x^{19} + x^{17} + x^{13} - x^{11}$;

   (c) x^{19};

   (d) $-x^{19} + x^{18} + x^{17} - x^{16}$.

6. Let V be the vector space of all polynomials in t of degree less than or equal to n. The following are bases of V:

   (a) $\{1, t, \ldots, t^n\}$;

   (b) $\{1, t-1, (t-1)^2, \ldots, (t-1)^n\}$;

   (c) $\{1+t, t+t^2, t^2+t^3, \ldots, t^{n-1}+t^n\}$.

7. Calculate the characteristic polynomial of $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

   (a) $-\lambda^3 - \lambda^2 + 12\lambda$.

   (b) $\lambda^3 - \lambda^2 - 12\lambda$.

   (c) $3\lambda^3 - 6\lambda^2 + 12\lambda$.

   (d) $4\lambda^3 + \lambda^2 - 12\lambda$.

8. Let V be the space of polinomial of degree les than or equal to 2 with coefficients in $\mathbb{R}$, equipped with the inner product:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Which of the following are orthonormal bases for V?

   (a) $1, t, t^2$;
   (b) $(t), \cos(t)$;
   (c) $1, 2\sqrt{3}(t - \frac{1}{2}), 6\sqrt{5}(t^2 - t + \frac{1}{6})$;
   (d) $1, (t - \frac{1}{2}), (t^2 - t + \frac{1}{6})$.

9. Let V be the set of all infinite sequences $(a_1, a_2, a_3, \ldots)$ of real numbers with the property that $a_i = a_{i-2} + a_{i-1}$ for $i \geq 3$. Then

   (a) V is not a vector space over $\mathbb{R}$.
   (b) V is a vector space over $\mathbb{R}$ of dimension 2.
   (c) V is a vector space over $\mathbb{R}$ of dimension 3.
   (d) V is uan vector space over $\mathbb{R}$ of infinite dimension.

10. Sean A and B two real $n \times n$ matrices. Let $\operatorname{tr}(A)$ be the trace of the matrix A. Which of the following statements always hold?

   (a) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
   (b) $\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$;
   (c) $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$;
   (d) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

11. Let A be an $n \times n$ matriz and λ an eigenvalue of A with eigenvector v. Which of the following statements is true?

   (a) $-v$ is an eigenvector of $-A$ with eigenvalue $-\lambda$.
   (b) If B is an $n \times n$ matrix and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.
   (c) Let c be a scalar. Then $(\lambda + c)^2$ is an eigenvalue of $A^2 + 2cA + c^2 I$.
   (d) If μ is an eigenvalue of an $n \times n$ matrix B, then $\lambda + \mu$ is an eigenvalue of $A + B$.

12. The following are eigenvalues of the matrix

$$A = \begin{pmatrix} 10001 & 3 & 5 & 7 & 9 & 11 \\ 1 & 10003 & 5 & 7 & 9 & 11 \\ 1 & 3 & 10005 & 7 & 9 & 11 \\ 1 & 3 & 5 & 10007 & 9 & 11 \\ 1 & 3 & 5 & 7 & 10009 & 11 \\ 1 & 3 & 5 & 7 & 9 & 10011 \end{pmatrix}.$$

(a) 0 and 1;

(b) 1000 and 1036;

(c) 10000 and 10036;

(d) 1, 10, 100, 1000, 10000 and 100000.

13. Which of the following series converge?

$$\text{I.- } \sum_{m=1}^{\infty} \frac{\ln(m^{-3})}{m^{-3}};$$

$$\text{II.- } \sum_{m=1}^{\infty} \frac{\ln(3)}{3m};$$

$$\text{III.- } \sum_{m=1}^{\infty} \frac{m}{3^m}.$$

(a) None;

(b) I;

(c) II;

(d) III.

14. Which are subsequences of the sequence $\{a_n\}$ defined by

$$a_n = (-1)^n \left(1 + \frac{1}{n}\right)?$$

$$\begin{aligned} &\text{I.- } b_n \text{ where } b_n = 1 + \frac{1}{n}; \\ &\text{II.- } c_n \text{ where } c_n = 1 + \frac{1}{2n}; \\ &\text{III.- } d_n \text{ where } d_n = 1 + \frac{1}{2n-1}. \end{aligned}$$

(a) None;
(b) I;
(c) II;
(d) III.

15. Calculate

$$\lim_{x \to 0} \frac{x + \ln(1-x)}{x - \ln(1+x)}.$$

(a) 0;
(b) -1;
(c) 2;
(d) 1.

16. Calculate

$$\lim_{h \to 0} \frac{\int_1^{x+h} e^{-t^2} dt - \int_1^x e^{-t^2} dt}{h}.$$

(a) $-2xe^{-x^2}$;
(b) 0;
(c) 1;
(d) e^{-x^2};

17. Let $u = \sqrt{x^2+9}$ and $v = 3x^2 - 2x$. Calculate du/dv as a function of x.

(a) $\frac{1}{4(3x-1)\sqrt{x^2+9}}$ with $x \neq 1/3$;
(b) $\frac{3x-1}{2\sqrt{x^2+9}}$;
(c) $\frac{2x(3x-1)}{\sqrt{x^2+9}}$;
(d) $\frac{x}{2(3x-1)\sqrt{x^2+9}}$ with $x \neq 1/3$.

18. What is the slope of the straight line tangent to the curve $y^3 - x^2y + 6 = 0$ at the point $(1, -2)$?

    (a) $-2/5$;
    (b) $-4/11$;
    (c) $4/11$;
    (d) 8.

19. Let $f(x, y) = x^3 + 6xy + y^3 + 3$. Which of the following are relative maxima of f?

    (a) $(0, 0)$;
    (b) $(-2, -2)$;
    (c) $(2, -2)$;
    (d) $(2, 2)$.

20. Which of the following characterize a solution of the differential equation

$$y \ln y + xy' = 0 \text{ with } x > 0?$$

    (a) $x \ln y = 1$;
    (b) $xy \ln y = 1$;
    (c) $(\ln y)^2 = 2$;
    (d) $-y(\ln y)(\ln x) = 1$.

21. Let f be a continuous function in the closed interval $[0, 1]$. Which of the following statements is true?

$$\text{I.- } \int_0^1 f(x^2)dx = \int_0^1 (f(x))^2 dx$$

$$\text{II.- } \int_0^1 f\left(\frac{x}{2}\right) dx = 2\int_0^1 f(x)dx$$

$$\text{III.- } \left(\int_0^1 f(x)dx\right)^2 = \int_0^1 (f(x))^2 dx$$

    (a) None;
    (b) I;
    (c) II;
    (d) III.

22. Consider the curve en $\mathbb{R}^3$ given by $(\cos t, \cos t, \sqrt{2}\sin t)$. What is the unit tangent vector at $t = \pi/3$?

   (a) $(-\sqrt{3}/2, -\sqrt{3}/2, \sqrt{3}/2)$;
   (b) $(-\sqrt{6}/4, -\sqrt{6}/4, 1/2)$;
   (c) $(1/2, 1/2, \sqrt{2}/2)$;
   (d) $(\sqrt{3}/2, \sqrt{3}/2, \sqrt{2}/2)$.

23. Let S be the closed region in the first quadrant of the plane bounded by $x^2+y^2 = 9$, the y-axis, and the x-axis. Calculate

$$\int\int_S xy(x+y)\, dA.$$

   (a) $27/2$;
   (b) 27;
   (c) $162/5$;
   (d) $324/5$.

24. Let $f(x, y) = xe^y/(xy+3)$ for $xy > 0$. Calculate $\frac{\partial f}{\partial x}$.

   (a) e^y/y;
   (b) $-3e^y/(xy+3)^2$;
   (c) $3e^y/(xy+3)^2$;
   (d) $(2xye^y + 3e^y)/(xy+3)^2$.

25. The differentiable closed curve C in the complex plane does not pass through any real integer. With this information, the line integral

$$\int_C \frac{dz}{(z-1)(z-2)(z-3)}$$

   (a) Has an infinite number of possible values.
   (b) Could be infinite.
   (c) Has exactly 3 possible values.
   (d) Only takes imaginary values.

26. Let G be a group and let H_1 and H_2 be two subgroups of G such that $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$. Then

   (a) $H_1 \cup H_2$ is never a subgroup of G.
   (b) $H_1 \cup H_2$ is always a subgrupo of G.
   (c) $H_1 \cup H_2$ may be a subgrupo of G.

27. Every group of order 24 satisfies the following.

   (a) It has a normal subgroup of order 4 or 8.
   (b) It has a normal subgroup of order 4 and a normal subgroup of order 8.
   (c) It has a normal subgroup of order 4.
   (d) It has a normal subgroup of order 8.

28. Let a, b, c, d be complex numbers and let $f(z) = (az + b)/(cz + d)$. Then

   (a) f has at least one fixed point in the complex plane.
   (b) f can have exactly 3 fixed points in the complex plane.
   (c) If f is not constant, then $f(z)$ may take any complex value, of one defines $f(\infty) = a/c$.
   (d) If f sends a circle A to a circle B, then f sends the interior of A to the interior of B.

29. Let $X = \mathbb{N} \times \mathbb{Q}$, a topological subspace of $\mathbb{R}^2$, and $P = \{(n, \frac{1}{n}) : n \in \mathbb{N}, n > 0\}$. Then in the space X

   (a) P is closed but not open.
   (b) P is abierto but not closed.
   (c) P is open and closed.
   (d) P is not open nor closed.

30. Mark all the options for which the central theorem of calculus holds: $F(x) - F(0) = \int_0^x F'(t)dt$

   (a) F is continuous.
   (b) F is piecewise linear.
   (c) F is absolutely continuous.
   (d) F is continuous and of bounded variation.