Centre for Research and Advanced Study at IPN

Department of Mathematics

Master' Degree Program Admission Examination

July 9, 1999

1. Linear Algebra

1.1 Consider a matrix of order n with real entries and I an identity matrix of order n. If A2 = 2I, prove that A + I is reversible and express its reverse in terms of A and I.

1.2 Determine the matrix (respect to the canonical basis) of a linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that satisfies $T^2 = I$ y $T((1,1)) = (1,0)$..

1.3 Find the values of the following vectors that generate a subspace of dimension 2.

$$\alpha_1 = (a,1,0), \alpha_2 = (1,a,1), \alpha_3 = (0,1,a), \alpha_4 = (1,1,1).$$

2. Calculus

2.1 Calculate the derivative of function F defined at [0, 1] as:

$$f(x) = \int_{x^2}^{x} \sqrt{1+t^2 dt.}$$

2.2 Prove that one of the following series is convergent and the other one divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad y \qquad \sum_{n=1}^{\infty} \frac{2}{n^2}$$

2.3 Let k be a fixed positive integer and a real number such that $0 < a < 1.$
Prove that: $\lim\limits_{n \longrightarrow \infty} \binom{n}{k} a^n = 0$

Remember that, by definition, $\binom{n}{k} = \frac{n!}{k!(n-k)}.$

3. Optional problems

3.1 Let $\mathbb{R}$ be a set of real numbers with a usual topology. Which of the following statements are correct?

a) The union of each finite family of open sets is an open set.
b) The union of each family of closed sets is a closed set.
c) Each bounded and infinite set has a succession of different points that converge at R.

3.2 Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ y $g(x) = b_m x^m + \ldots + b_1 x + b_0$ be two polynomial functions with real quotients. Prove that if $f(a) = g(a)$ for each $a \in [0, 1]$, then $f(a) = g(a)$ for each $a \in \mathbb{R}$.

3.3 Let f, g be two continuous and non-negative functions on $[a, \infty)$ and assume that the following limit exists:

$$L := \lim_{x \longrightarrow \infty} [f(x)/g(x)].$$

Demonstrate that:

a) If $0 < L < \infty$ then both integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or both diverge.
b) If L = 0 and $\int_a^\infty g(x)dx$ converge then $\int_a^\infty f(x)dx$ converges.
c) If $L = \infty$ and $\int_a^\infty g(x)dx$ diverge, then $\int_a^\infty f(x)dx$ diverges.

3.4 If (X, d) is a metric space and A is a non-empty subset of X, we define:

$$d(x, A) := \inf\{d(x, a) | a \in A\}$$

If $\overline{A}$ denotes the lock of A, demonstrate that $\overline{A} = \{x | d(x, A) = 0\}$.

3.5 Let $f : [0, 1] \longrightarrow (0, 1)$ be a continuous function. Consider the equation $g(x) = 1$, where

$$g(x) = 2x - \int_0^x f(t)dt.$$

a) Does this equation have any solution on [0, 1]?
b) Does this equation have a unique solution on [0, 1]?