Centre for Research and Advanced Study at IPN

Department of Mathematics

Master's Degree Program Admission Examination

January 18, 1999

1. Linear Algebra

1.1 Let W be the subspace of $\mathbb{R}^3$ generated by $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where
$\alpha_1 = (2,1,1), \alpha_2 = (-1,2,0), \alpha_3 = (7,-4,2), \alpha_4 = (1,1,1).$

Determine a basis for W contained in $\mathcal{B}$.

1.2 Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(x,y) = (-x+y, x+2y)$.

a) Find the matrix representation A of T with respect of the canonical basis of $\mathbb{R}^2$.
b) Let $\mathcal{B} = \{\beta_1, \beta_2\}$, where $\beta_1 = (1,0), \beta_2 = (1,1)$. Find the matrix representation D of T relative to the basis $\mathcal{B}$.
c) Find (or demonstrates that it exits) a reversible matrix P such that

$$D = P^{-1}AP.$$

1.3 Let A be a square matrix with real entries. It is said that A is diagonal if a reversible matrix P exists with real entries such that PAP^{-1} is a diagonal matrix.

a) Provide a condition necessary or sufficient so that A is diagonalizable.
b) Let be

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Demonstrate that A is not diagonalizable.

c) Demonstrate that if $A^2 = A$, then A is diagonalizable.

2. Calculus

2.1 Calculate the derivative of the function F defined in [0, 1] as
$F(x) = \in 3{,}2t_{x^2}^{x} \sqrt{1+t^2}dt.$

2.2 Prove that some real number $a > 0$ exists such that $\tan a = a.$

2.3 If $f(0,0) = 0,$ and $f(x,y) = \frac{xy}{x^2+y^2}$ si $(x,y) \neq (0,0)$

prove that partial derivatives of f exist in each point of $\mathbb{R}^2$. Is f continuous in (0, 0)?

3. Optional Problems

3.1 Let $\mathbb{R}$ be the set of real number with usual topology. Which of the following statements are true?

a) The finite union of open sets is open
b) The arbitrary union of closed sets is closed.
c) Each infinite and bound set has a succession of distinct points that converge in $\mathbb{R}$

3.2 Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ and $g(x) = b_m x^m + \ldots + b_1 x + b_0$ be two polynomial functions with real coefficients. Prove that if $f(a) = g(a)$ for each $a \in [0,1],$ then $f(a) = g(a)$ for each $a \in \mathbb{R}$.

3.3 Let $f(a)$ be two continuous and non-negative functions on $[a, \infty)$ and let be

$$L := \lim_{x \to \infty} [f(x)/g(x)]$$

Demonstrate:

a) If $0 < L < \infty,$ then both integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or both diverge.
b) If $L = 0$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

c) If $L = \infty$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

3.4 If (X, d) is a metric space and A is a non-mepty subset of X, we define
$d(x, A) = \inf \{d(x, a) | a \in A\}$.

Si $\overline{A}$ denotes the lock for A, demonstrate that $\overline{A} = \{x | d(x, A) = 0\}$.