Research and Advanced Studies of IPN
Department of Mathematics

Examination of admission to the MSc

July 1, 2011

1. **Linear Algebra**
   1.1. Suppose that *A* and *B* are endomorphisms of a vector space *V* of finite dimension over a field *F*. Prove or give a counterexample to the following statements:
      a) Any eigenvector of *AB* is an eigenvector of *BA*.
      b) Every eigenvalue of *AB* is an eigenvalue of *BA*.

   1.2. Prove that every vector space (not necessarily finite dimensional) has a basis.

   1.3. Let *A* be an $n \times n$ matrix with entries in the integers. Prove that there exists a matrix *B* with entries in the integers such that $AB = I_n$ if and only if $|\det A| = 1$. Where I_n is the identity matrix of size $n \times n$.

2. **Calculus**
   2.1. Show that $\frac{(x^2+y^2)}{4} \leq e^{(x+y-2)}$ for all $x \geq 0, y \geq 0$.

   2.2. Let $x_1, x_2, \ldots$ is a sequence of nonnegative real numbers such that $x_{n+1} \leq x_n + \frac{1}{n^2}$ for all $n \geq 1$. Show that $\lim_{n\to\infty} x_n$ exists.

   2.3. Prove that $\int_0^{\pi} \frac{x \sin(x)}{1+\cos^2(x)} dx = \frac{\pi^2}{4}$

3. **Optional problems**
   3.1. Let *T* be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Prove that there exists an $m \in \mathbb{R}$ such that $|T(v)| \leq m|v|$, for all $v \in \mathbb{R}^m$.

   3.2. Show that in $\mathbb{R}^n$ a set is compact if and only if it is closed and bounded. Is it true this result in any metric space?

   3.3. Let *G* be a finite group such that $|G| = p^2$, with *p* a prime. Prove that *G* is abelian.