Centro de Investigación y de Estudios Avanzados del IPN
Departament of Mathemtics

Admissions Examination for the Master's Program

2 July 2012

Instructions: Solve all problems of sections 1 and 2 and as many as possible from section 3. All solutions must be justified appropiately. The examination will last for three hours..

I. **Linear algebra**

1.1 Let A be an $n \times n$ matrix with entries in the set $\{0, 1\}$ with exactly two ones in each column and two ones in each row. Give necessary and sufficient conditions for the rank of A to be n.

1.2 Let a and b be two real numbers. Find the determinant of the $n \times n$ matrix whose entries are:

$$a_{ij} = \begin{cases} a & \text{if } i \neq j \\ a+b & \text{if } i = j \end{cases} \tag{1}$$

for $1 \leq i \leq n$ and $1 \leq j \leq n$.

1.3 Determine whether the series $\sum_{i=1}^{m} \frac{1}{i^2} A^i$ converges where

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

2. Calculus

2.1 Determine whether the following function is differentiable:

$$f(x,y) = \begin{cases} 2xy\frac{x^2-y^2}{x^2+y^2} & \text{si } x^2+y^2 \neq 0 \\ 0 & \text{en otro caso} \end{cases}$$

2.2 Prove that

$$\int_0^{\pi} \frac{x sen(x)}{1 + cos^2(x)} dx = \frac{\pi^2}{4}.$$

2.3 Give an example of a sequence x_n of real numbers satisfying the following conditions (if such exists):

(a) x_n converges and $\lim_{n\to\infty}(x_n)^{\frac{1}{n}} = 1$.

(b) x_n diverges and $\lim_{n\to\infty}(x_n)^{\frac{1}{n}} = 1$.

(c) x_n diverges, is bounded, and $\lim_{n\to\infty}(x_n)^{\frac{1}{n}}$ exists.

3. Optional problems

3.1 Prove that in $\mathbb{R}^n$ a set is compact if and only it is closed and bounded. Is this result true in any metric space?.

3.2 Find $\lim_{x\to 0}(\lim_{n\to\infty}\frac{1}{x}(I - A^{2n}))$, where

$$A = \begin{pmatrix} 0 & -(\frac{x}{n})^{\frac{1}{2}} \\ (\frac{x}{n})^{\frac{1}{2}} & -1 \end{pmatrix}$$

3.3 Give an example of a function which is continuous on the irrationals and discontinuous on the rationals. Justify.

3.4 Let G be an abelian group with finitely many subgroups. Prove that G is finite.