Centro de Investigación y de Estudios Avanzados del IPN
Department of Mathematics

Examination for the Master's Program

7 January 2013

Instructions: Solve all problems of sections 1 and 2 and as many as possible from section 3. All solutions must be justified appropriately. The examination will last for three hours.

I. **Linear algebra**

1.1 Let A be an $n \times n$ matrix with entries in the set $\{0, 1\}$ with exactly two ones in each column and two ones in each row. Give necessary and sufficient conditions for the rank of A to be n.

1.2 Let A,B be $n \times m$ and $m \times n$ matrices respectively. If $AB = I_n$ and $BA = I_m$, prove that $n = m$, where I_k is the identity matrix of order $k \times k$.

1.3 Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Prove there exists an M such that $|T(x)| \leq M|x|$ for all $x \in \mathbb{R}^n$.

2. Calculus

2.1 Find the limit of the sequence $\frac{(n!)^{\frac{1}{n}}}{n}$.

2.2 Let f be a differentiable function such that f' is continuous in the interval $[a, b]$. Prove that

$$\int_a^b f'(t)dt = f(b) - f(a).$$

2.3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = |xy|^{\frac{1}{2}}$. Prove that f is not differentiable at $(0, 0)$.

3. Problemas opcionales

3.1 Find all positive integers n for which n^2 divides $2^n + 1$.

3.2 Prove that a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Is this result valid in an arbitrary metric space?

3.3 Does there exist a function which is discontinuous on the irrationals and continuous on the rationals? Justify.