Why Critical Ideals?

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Mathematical Congress of the Americas 2013
August 5-9, 2013 Guanajuato, Mexico

Outline

1. Critical ideals of a graph
   - Critical groups and characteristic polynomial
2. The algebraic co-rank
3. Critical ideals of graphs with twins
   - Duplication
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   - Some Conjectures
   - Perspectives
4. Graphs with a small algebraic co-rank
   - $\gamma = 2$
   - $\gamma = 3$
   - Some Conjectures
5. Trees
   - Some Conjectures

The generalized Laplacian matrix

Let G be a signed multidigraph, $\mathcal{P}$ a Principal Ideal Domain (PID), and

$$X_G = \{x_u \mid u \in V(G)\}$$

the set of variables indexed by the vertices of G.

Definition

The *generalized Laplacian matrix* of G is given by

$$L(G,X)_{u,v} = \begin{cases} x_u & \text{if } u = v, \\ -\sigma(uv)m_{uv}1_{\mathcal{P}} & \text{otherwise,} \end{cases}$$

where m_{uv} is the number of arcs leaving u and entering to v and $1_{\mathcal{P}}$ is the unity of $\mathcal{P}$.

The critical ideals of a graph

Furthermore, if $\mathcal{P}[X_G]$ is the polynomial ring over $\mathcal{P}$ on X_G.

Definition

The *critical ideals* of G are the determinantal ideals given by

$$I_i(G,X) = \langle \mathrm{minors}_i(L(G,X)) \rangle \subseteq \mathcal{P}[X_G] \text{ for all } 1 \leq i \leq n,$$

where $\mathrm{minors}_i(L(G,X))$ are the minors of $L(G,X)$ of size i.

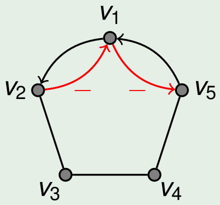
Clearly

$$\langle 0 \rangle \subsetneq I_n(G) \subseteq \cdots \subseteq I_2(G) \subseteq I_1(G) \subseteq \langle 1 \rangle.$$

Moreover, if H is an induced subdigraph of G, then

$$I_k(H) \subseteq I_k(G) \text{ for all } 1 \leq k \leq |V(G)|.$$

Example



$$L(G,X)=\begin{bmatrix} x_1 & -1 & 0 & 0 & 1 \\ 1 & x_2 & -1 & 0 & 0 \\ 0 & -1 & x_3 & -1 & 0 \\ 0 & 0 & -1 & x_4 & -1 \\ -1 & 0 & 0 & -1 & x_5 \end{bmatrix}$$

If $\mathcal{P}=\mathbb{Z}$, then $I_i(G,X)=\langle 1\rangle$ for all $i\leq 3$,

$$I_4(G,X)=\langle x_1x_2+x_4+1, x_2x_3-x_5-1, x_3x_4+x_1-1, x_4x_5-x_2-1, x_1x_5+x_3+1\rangle,$$

and

$$I_4(G,X)=\langle x_1x_2x_3x_4x_5-x_1x_2x_3+x_2x_3x_4-x_1x_2x_5-x_1x_4x_5+x_3x_4x_5+x_1-x_2-x_3-x_4-x_5-2\rangle.$$

Correspondence

If $\mathcal{P} = \mathbb{Z}$ and $x_i = d_G(v_i)$, then $L(G, X)$ becomes the usual Laplacian matrix. The critical group of G is given by

$$K(G) \oplus \mathbb{Z} = \mathbb{Z}^V / \mathrm{Im}\, L(G)^t.$$

Theorem

If $K(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}_{f_i}$ with $f_1 | \cdots | f_{n-1}$, then

$$I_i(G, X)_{X = d_G(G)} = \langle \prod_{j=1}^{i} f_j \rangle \text{ for all } 1 \leq i \leq n-1.$$

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Theorem

Let G be a graph and v a vertex of G such that $d_G^+(u) = d_G^-(u)$ for all $u \in V(G)$. If $K(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}_{f_i}$ with $f_1 | \cdots | f_{n-1}$, then

$$I_i(G \setminus v, X)_{X = d_G(u)} = \langle \prod_{j=1}^{i} f_j \rangle \text{ for all } 1 \leq i \leq n-1.$$

If $\mathcal{P}$ is a field, then the critical ideals of G are principal. That is, there exist $p_i(t) \in \mathcal{P}[t]$ for all $1 \leq i \leq n$ such that

$$I_i(G,t) = \langle \prod_{j=1}^{i} p_j(t) \rangle \text{ for all } 1 \leq i \leq n.$$

Example

If G is the complete graph with six vertices minus a perfect matching and $\mathcal{P} = \mathbb{Q}$, then

$$I_i(G,t) = \begin{cases} \langle 1 \rangle & \text{if } 1 \leq i \leq 4, \\ \langle t^2(t+2) \rangle & \text{if } i = 5, \\ \langle t^3(t+2)^2(t-4) \rangle & \text{if } i = 6. \end{cases}$$

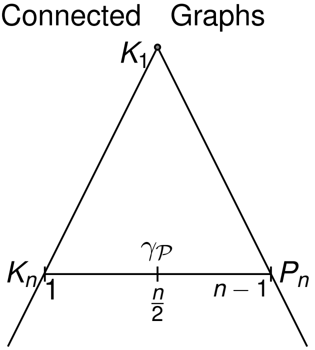
Therefore we get a factorization of the characteristic polynomial of G.

The algebraic co-rank

Definition

Given a graph G and $\mathcal{P}$ a PID, let

$$\gamma_{\mathcal{P}}(G) = \max\{i \mid I_i(G, X) = \langle 1 \rangle\}.$$



$$0 \leq \gamma_{\mathcal{P}}(G) \leq |V(G)| - 1.$$

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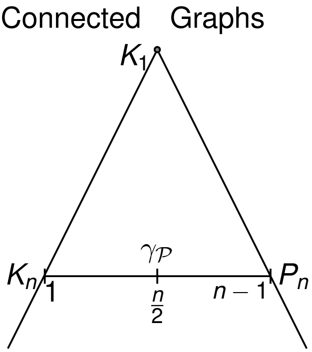
$\gamma_{\mathcal{P}}(G) = 0 \Leftrightarrow G = K_1$.



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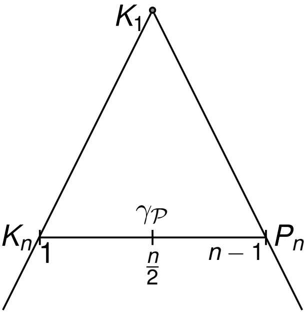
Theorem (Lorenzini)

$\gamma_{\mathcal{P}}(G) = 1 \Leftrightarrow G = K_n.$

Connected Graphs



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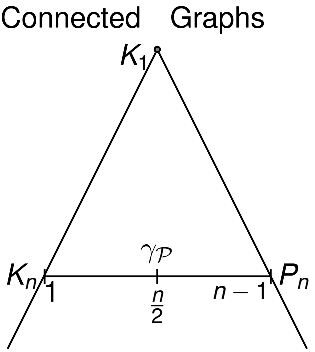
Theorem

$\gamma_{\mathcal{P}}(G) = n - 1 \Leftrightarrow G = P_n.$



$0 \leq \gamma_{\mathcal{P}}(G) \leq |V(G)| - 1.$

Critical ideals of graphs with twins

Definition

Let G and $v \in V(G)$, the *duplication*, denoted by $d(G, v)$, is the graph given by $V(d(G, v)) = V(G) \cup \{v^1\}$ and

$$E(d(G, v)) = \{v^1 u \mid u \in N_+(v)\} \cup \{uv^1 \mid u \in N_-(v)\}$$

In this case we say that v and v^1 are *false twins*.

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Definition

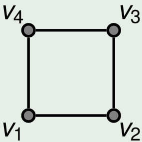
The *replication* of v on G, denoted by $r(G, v)$, is the graph obtained from $d(G, v)$ by adding the arcs vv^1 and $v^1 v$.

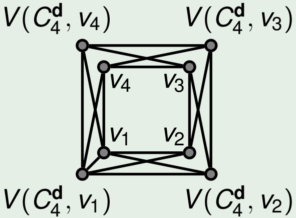
In this case we say that v and v^1 are *true twins*.

Critical ideals of graphs with twins

Example

Let C_4 and $\mathbf{d} = (-1, 1, 1, 1)$, then $C_n^{\mathbf{d}}$





Theorem

Let G be a graph with $n \geq 2$ vertices and $\mathbf{d} \in \mathbb{Z}^n$. Then

$$I_j(G^{\mathbf{d}}, X) \subseteq \langle \{x_v, \ldots, x_{v^{\mathbf{d}_v}} \,|\, \mathbf{d}_v \geq 1\}, \{x_v + 1, \ldots, x_{v^{-\mathbf{d}_v}} + 1 \,|\, \mathbf{d}_v \leq -1\}, I_j(G, X)_{\{x_v = -1 \,|\, \mathbf{d}_v \leq -1\} \cup \{x_v = 0 \,|\, \mathbf{d}_v \geq 1\}} \rangle$$

Moreover, $I_j(G^{\mathbf{d}}, X) = \langle 1 \rangle \Leftrightarrow I_j(G, X)_{\{x_v=-1|\mathbf{d}_v \leq -1\} \cup \{x_v = 0|\mathbf{d}_v \geq 1\}} = \langle 1 \rangle$.

Theorem

Let G be a graph with $n \geq 2$ vertices and $\mathbf{d} \in \mathbb{Z}^n$. Then

$$I_j(G^{\mathbf{d}}, X) \subseteq \langle \{x_v, \ldots, x_{v\mathbf{d}_v} \mid \mathbf{d}_v \geq 1\}, \{x_v + 1, \ldots, x_{v-\mathbf{d}_v} + 1 \mid \mathbf{d}_v \leq -1\}, I_j(G, X)_{\{x_v=-1 \mid \mathbf{d}_v \leq -1\} \cup \{x_v=0 \mid \mathbf{d}_v \geq 1\}} \rangle$$

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Corollary

Let G graph with n vertices and $\delta \in \{0, 1, -1\}^n$, then

$$\gamma_{\mathcal{P}}(G^{\mathbf{d}}) = \gamma_{\mathcal{P}}(G^{\delta}) \text{ for all } \mathbf{d} \in \mathbb{Z}^n \text{ with } \mathrm{supp}(\mathbf{d}) = \delta.$$

Moreover, $\gamma_{\mathcal{P}}(G^{\mathbf{d}}) \leq n$ *for all* $\mathbf{d} \in \mathbb{Z}^n$.

This bound is tight.

Duplication

Let $P_{i,k} = \{\prod_{l=1}^{i} x_{v_l^1}\}$, $P_{0,k} = \{1\}$ and for all $d, d' \geq 0$ let

$$\lambda(d, d') = \begin{cases} 0 & \text{if } d, d' = 0, \\ 1 & \text{otherwise.} \end{cases}$$

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Theorem

Let G be a graph, $v \in V(G)$, $g = \gamma_{\mathcal{P}}(G)$, $g' = \gamma_{\mathcal{P}}(d(G,v))$, and $d = g - \gamma_{\mathcal{P}}(G \setminus v)$, $d' = g' - g$. If $g \geq 1$, then $0 \leq d + d' \leq 2$ and

$$I_{g'+k}(d^{k+\lambda+i}(G,v), X) = \langle P_{k,k+\lambda+i}, \left\{P_{l,k+\lambda+i} \cdot I_{g'+k-l}(G,X)_{x_v=0}\right\}_{l=0}^{k-1}\rangle$$

for all $k \geq 1$ and $i \geq 0$.

Replication

Let $\widetilde{P}_{i,k} = \{\prod_{l=1}^{i}(x_{v_l^1} + 1)\}$, $\widetilde{P}_{0,k} = \{1\}$.

Theorem

Let G be a graph, $v \in V(G)$, $g = \gamma_{\mathcal{P}}(G)$, $g' = \gamma_{\mathcal{P}}(r(G,v))$, and $d = g - \gamma_{\mathcal{P}}(G \setminus v)$, $d' = g' - g$. If $g \geq 1$, then $0 \leq d + d' \leq 2$ and

$$I_{g'+k}(r^{k+\lambda+i}(G,v),X) = \langle \widetilde{P}_{k,k+\lambda+i}, \left\{ \widetilde{P}_{l,k+\lambda+i} \cdot I_{g'+k-l}(G,X)_{x_v=-1} \right\}_{l=0}^{k-1} \rangle$$

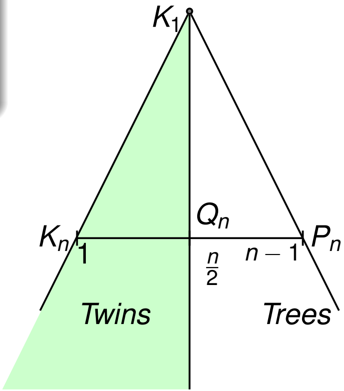
for all $k \geq 1$ and $i \geq 0$.

Some Conjectures

Conjecture

If $\gamma_{\mathcal{P}}(G \setminus v) = \gamma_{\mathcal{P}}(G)$ for all $v \in V(G)$, then G has at least a pair of twin vertices.

Connected Graphs



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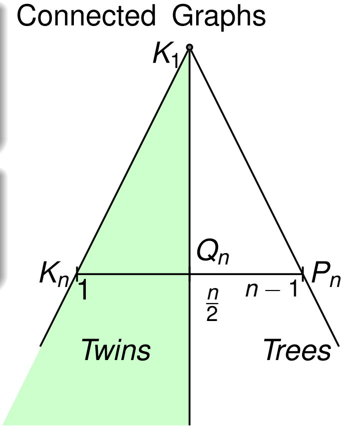

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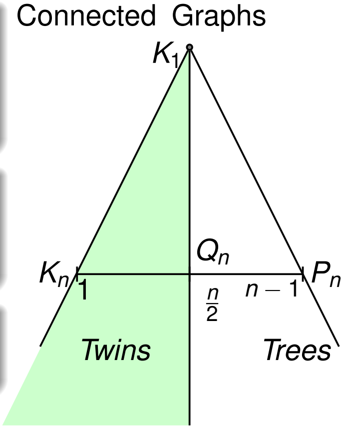
If G is twin-free, then $\gamma_{\mathcal{P}}(G) \geq \frac{n}{2}$.



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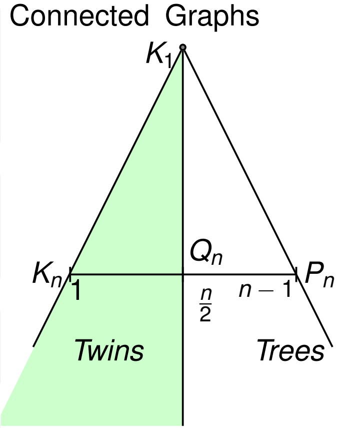
Remark

1) $\Rightarrow$ 2) $\Leftrightarrow$ 3).



Perspectives

We also have calculated the critical ideals of

1. Complete multipartite graphs,
2. Threshold graphs,
3. Cographs.

We are interested

1. Quasi-threshold graphs,
2. Chordal graphs,
3. Split graphs,
4. Hypercubes,
5. Bipartite graphs.

Graphs with a small algebraic co-rank

Definition

Let $\Gamma_{\leq k} = \{G \,|\, G$ is a simple connected graph with $\gamma(G) \leq k\}$.

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Definition

A graph is called $\gamma_{\mathcal{P}}$-critical if $\gamma_{\mathcal{P}}(G \setminus v) < \gamma_{\mathcal{P}}(G)$ for all $v \in V(G)$.

Let $\mathrm{For}(\Gamma_{\leq k}) = \{G | \gamma_{\mathcal{P}}\text{-critical with } \gamma_{\mathcal{P}}(G) = k+1\}$.

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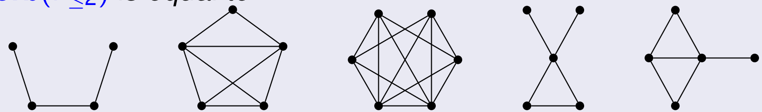
Remark

$G \in \Gamma_{\leq k}$ if and only if G is $\mathrm{For}_k(\Gamma_{\leq k})$-free.

Theorem

Forb$(\Gamma_{\leq 2})$ *is equal to*



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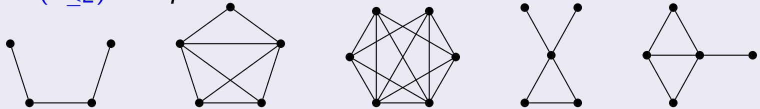
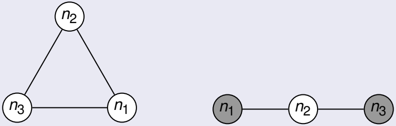


Theorem

Let G be a simple connected graph. Then, $G \in \Gamma_{\leq 2}$ if and only if G is an induced subgraph of one of the following graphs:



K_{n_1,n_2,n_3} L_{n_1,n_2,n_3}

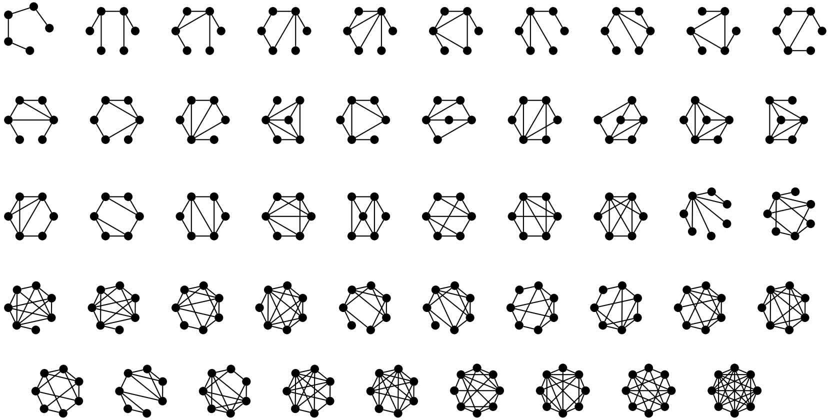


Figure: Family $\mathcal{F}_3$

Some Conjectures

Conjecture

For all $k \in \mathbb{N}$ the set **Forb**$(\Gamma_{\leq k})$ *is finite.*

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Conjecture

For all $k \in \mathbb{N}$ the set $\mathbf{Forb}(\Gamma_{\leq k})$ is finite.

Conjecture

There exists a finite set

$$\mathcal{H} = \{(G, \delta) \mid G \text{ is a graph and } \delta \in \{0,1\}^n\}$$

such that $H \in \Gamma_{\leq k}$ if and only if $H = G^{\mathbf{d}}$ with $\mathrm{supp}(\mathbf{d}) = \delta$ for some pair $(G, \delta) \in \mathcal{H}$.

Trees

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Definition

A 2-matching M of G^{loop} is minimal if not exists a 2-matching M' such that

- $|M| = |M'|$,
- $loops(M') \subsetneq loops(M)$.

Minimal 2-matchings

Theorem

If T be a tree with n vertices, then

$$I_i(T,x) = \langle \{\det(L(\mathcal{M},X)) \mid \mathcal{M} \in \mathcal{V}_2^i(T)\} \rangle \text{ for all } 1 \leq i \leq n,$$

where $\mathcal{V}_2^i(T)$ is the set of minimal 2-matching of T^{loop} of size i.

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Corollary

Let T be a tree, then $\gamma(T) = \nu_2(T)$.

Some Conjectures

Conjecture

If T be a tree with n vertices, then

$$B = \{\det(L(\mathcal{M}, X)) \,|\, \mathcal{M} \in \mathcal{V}_2^i(T)\}$$

is a reduced Gröbner basis of $I_i(T, x)$ for all $1 \leq i \leq n$.

Until now we proved this conjecture for i equal to $n-1$ and $\nu_2(T)+1$.

Thank You

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