Ultrametric space in Teichmüller theory

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This in collaboration with Prof. Alberto Verjovsky.

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where the moduli space $\mathcal{M}_g$ is taken as a fundamental domain in the Teichmüller space T_g respect to the mapping class group.
An action for the theory is obtained by summing all the contributions:

$$Z := \sum_{g \in \mathbb{N}_0} Z_g$$

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$$Z = \int_{\mathcal{M}} \prod_{i=1}^{\infty} dc_i d\bar{c}_i |\tilde{F}(c)|^2 \det(1 - Z(c)^\dagger Z(c))^{-13}$$

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$$T_\infty := \overline{\bigcup_{g\in\mathbb{N}_0} T_g}$$

This is a separable space.

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In particular, heuristically, we can write the resulting string theory on T_∞ as a Quantum Field Theory of these fields:

$$Z = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \; e^{S(\varphi,\bar{\varphi})}$$

Deformation

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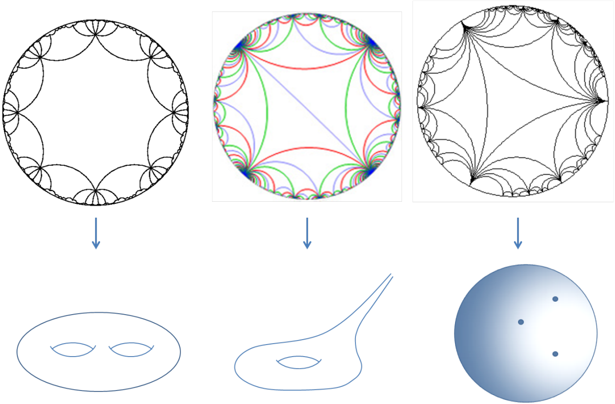
Consider the Poincaré-Koebe uniformization:

$$\Delta \to \Sigma$$

and the representation of $G := \pi_1(\Sigma)$ as a Fuchsian group:

$$\alpha : G \to Isom^+(\Delta)$$



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- μ *is a* G*-periodic differential if it is a differential and:*

$$\alpha(g)^* \mu = \mu \quad \forall\, g \in G$$

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Remark

The pullback of a differential in Σ *by the uniformization map is* G*-periodic differential in the Poincaré disk.*

How a Beltrami differential (deformation parameter) actually realizes a deformation?

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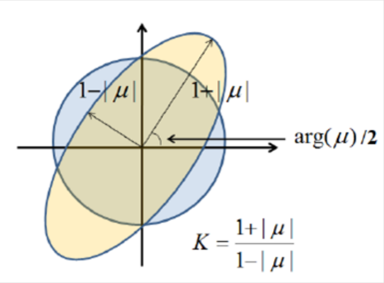
A Beltrami differential can be seen as an ∞-measurable field of ellipses:



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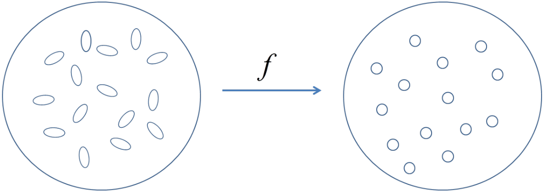
Is there a solution to this equation on the disk Δ?... Equivalently, Is there a map f on the disk straightening all the infinitesimal ellipses into infinitesimal circles?



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There are quasiconformal homeomorphisms solutions to the Ahlfors-Bers equation. Moreover, these solutions uniquely extends to a homeomorphism on the boundary and there is a unique solution f^{μ} *fixing* $1,\ i$ *and* -1.

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- *If* $\mu = 0$*, then* $\partial_{\bar{z}} f^\mu = 0$ *and by the Weil Lemma,* f^μ *is holomorphic.*
- *There are at most* $84(g-1)$ G*-equivariant biholomorphisms of the disk; i.e.* $|Aut(\Sigma_g)| \leq 84(g-1)$.

In particular, abusing of notation, for every $\mu \in L_\infty(\Sigma)_1$ we have a quasiconformal deformation $f^\mu : \Sigma \to \Sigma$.

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Equivalently, given a complex structure J, we define:

$$J^\mu := df^\mu \circ J \circ d(f^\mu)^{-1}$$

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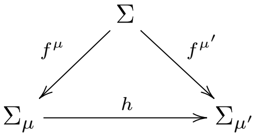

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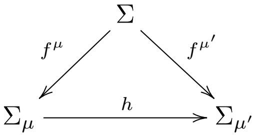
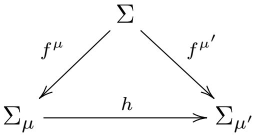


This relation gives the *coarse moduli space* $\mathcal{M}_g$ of compact Riemann surfaces of genus g.

To produce a *fine moduli* we strength the relation: We say that $\Sigma_\mu \overset{T}{\sim} \Sigma_{\mu'}$ if there is a homeomorphism h isotopic to the identity such that the following diagram commutes:



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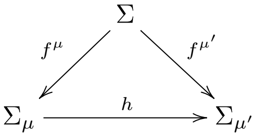

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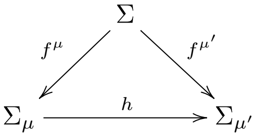

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$$\mathcal{M}(\Sigma) = T(\Sigma)/MCG(\Sigma)$$

$$MCG(\Sigma) := Homeo(\Sigma)/Homeo_0(\Sigma)$$

We define the *universal Teichmüller space* as follows:

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It is Universal in the sense that it contains all the finite dimensional Teichmüller spaces:

$$T(\Sigma) \subset T(1)$$

Universal Hyperbolic Lamination

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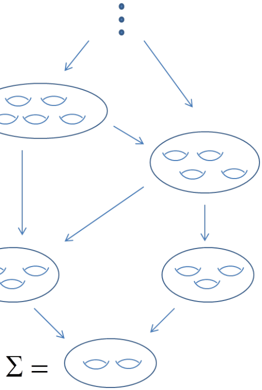
The diagonal action is equivariant respect to the Fuchsian representation α hence we have a finite holomorphic covering:

$$\begin{array}{ccc} G' \backslash G & \hookrightarrow & \Sigma_{G'} \\ & & \downarrow \\ & & \Sigma \end{array}$$

Because the construction is functorial, we actually have an inverse system of finite holomorphic coverings of Σ:



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The collection of finite index subgroups of G is a neighborhood system of the identity and by translation it defines a topology on G whose completion is the group G_∞ just defined. As a topological space, the group G_∞ is a compact totally disconnected Hausdorff space; i.e. a Cantor set.

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Because the group G is finitely generated, there must be a finite amount of subgroups of a given index hence the normal subgroups A_n are of finite index as well. Define the following valuation $val : G \to \mathbb{N} \cup \{\infty\}$ such that:

$$val(g) := max\{n \in \mathbb{N} \ \mid \ g \in A_n\}$$

if g is not the neutral element e and $val(e) := \infty$. Define the translation invariant metric d on the group G such that:

$$d(g,h) := e^{-val(g^{-1}h)}$$

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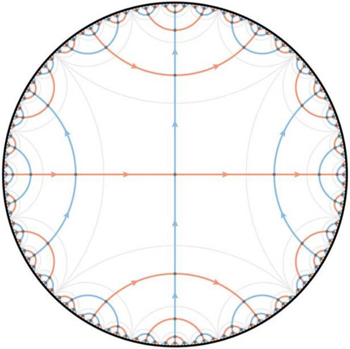
For example, consider the cusped torus. Its fundamental group is the free product $\mathbb{Z} * \mathbb{Z}$ and its Cayley graph is the following:



The inverse limit of the covering tower $(\Sigma_{G'})$ defined before is the *Universal Hyperbolic Lamination*:

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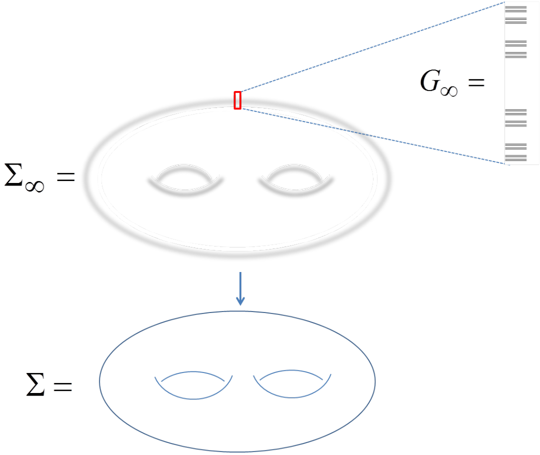
This is a lamination whose leaves are densely inmersed disks Δ. The leaf space is G_∞/G.

The following is a picture of the lamination:



Define the *baseleaf* $\iota : \Delta \to \Sigma_\infty$ as the composite map:

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Its basic open sets are:

$$V(U, G') := \bigcup_{g \in G'} \alpha(g)(U)$$

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The map $\iota : \Delta_{Emb} \hookrightarrow \Sigma_\infty$ is an embedding.

Because $\Delta \xrightarrow{id} \Delta_{Emb}$ is continuous, we have:

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Proposition

Consider a function $f : \Delta \to \mathbb{C}$. Then,

- *f is limit-periodic iff it is the uniform limit of periodic functions.*
- *f is limit-periodic iff there is a continuous function $g : \Sigma_\infty \to \mathbb{C}$ such that $\iota^* g = f$.*

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The following definition is due to D.Sullivan:

Definition

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Renormalized Weil-Petersson metric

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- If we consider nets of periodic differentials converging uniformly to the limitperiodic differentials respectively then:

$$\lim_{\substack{\longleftarrow \\ G'<G \\ [G':G]<\infty}} WP(\mu_{G'}, \nu_{G'}) = 0 \ or \ \infty$$

where WP is the usual Weil-Petersson metric.

We define the *renormailized Weil-Petersson metric*:

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Proposition

The renormailized Weil-Petersson metric is well defined; i.e. It converges in the space of limit-periodic differentials and is independent of the choice of the nets.

Remark

The renormalized Weil-Petersson metric is an extension of the usual one for G-periodic differentials.

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Actually, a physicist would think on this result as follows: The inverse limit of coverings is the *renormalization group* of the theory and the number of sheets of the covering is the renormalization factor of the respective energy level. Then, to get the measured observables on the respective energy level we have to quotient by the renormalization factor; i.e. by the index $[G', G]$. The limit gives the observable at fundamental scale.

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The complex analyitic Kähler coadjoint orbit:

$$Diff^{+}(S^{1})/Möb \hookrightarrow T(1)$$

is transversal to the Teichmüller space of the lamination in the universal one:

$$Diff^{+}(S^{1})/Möb \pitchfork T(\Sigma_{\infty})$$

Main results

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This can be seen as Kähler coordinates of the Teicmüller space of the lamination, labelled by an ultrametric space. The previous result is functorial; i.e. The following diagram commutes:

$$\begin{array}{ccc} C\left(G_\infty, T(\Sigma)\right) & \xrightarrow{\hat{f} \quad \simeq} & T(\Sigma_\infty) \\ \uparrow & & \uparrow \\ T(\Sigma)^n \simeq C\left(G' \backslash G, T(\Sigma)\right) & \xrightarrow{\simeq} & T(\Sigma_{G'}) \end{array}$$

The following is Theorem B:

Theorem

The $(g-1)$-times alternating product of the moduli space of genus two compact Riemann surfaces is a discrete fiber complex analytic Kähler covering of the moduli space of genus g compact Riemann surfaces:

$$Alt^{g-1}(\mathcal{M}_2) \twoheadrightarrow \mathcal{M}_g$$

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Thank you very much!!!