On p-adic string amplitudes in the limit when p approaches to one
(joint work with W. A. Zúñiga-Galindo)

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1. Mathematical approach for 'lim $p \to 1$'.

- p-adic string theory is related with ordinary string theory.
- Physicists have discovered that the limit when $p \to 1$ allows to pass from the non-Archimedean world to the Archimedean one.
- This limit doesn't have sense for the discrete variable p.
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- **Connections between p-adic string amplitudes and local zeta functions**.

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Results:

*p-adic string amplitudes are 'essentially' local zeta functions.

* Amplitudes are convergent integrals that admit meromorphic continuations as rational functions i.e. *the problem of regularization of p-adic string amplitudes.*

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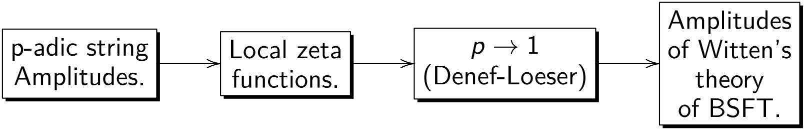

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- A brief introduction to multivariate local zeta functions.
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Multivariate local zeta functions

Let K be a non-Archimedean field, $f_i(\boldsymbol{x}) \in K[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, be a non-constant polynomial, $i = 1, \ldots, r$, $\Phi : K^n \longrightarrow \mathbb{C}$ be a locally constant function with compact support, and $s_i \in \mathbb{C}$ be complex numbers. The multivariate local zeta function attached to Φ, $\boldsymbol{s} = (s_1, \ldots, s_r)$, and $\boldsymbol{f} = (f_1, \ldots, f_r)$ is defined as

$$Z_\phi(\boldsymbol{s}; \boldsymbol{f}, K) = \int_{K^n} \phi(\boldsymbol{x}) \prod_{i=1}^{r} |f_i(\boldsymbol{x})|_K^{s_i} |d\boldsymbol{x}|_K.$$

- Loeser studied these integrals and showed that they admit meromorphic continuations as rational functions in the variables q^{-s_i}, $i = 1, \ldots, r$.
- q is the cardinality of the residue field $\overline{K}$ of K.

Loeser, F., *Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants.* Ann. Sc. Ec. Norm. Sup. 22 (1989), no. 3, 435-471.

The local zeta functions are related with:

- Number of solutions of polynomial congruences module p^m,
- Group and ring theory,
- Algebraic geometry,
- Singularity theory,
- and other areas of mathematics.
- **p-adic string amplitudes....?**

p-adic String Amplitudes

- String theory was motivated by need of understanding aspects of the strong interactions of elementary particles. Strong interactions are described by functions (amplitudes), which satisfy some physical requirements.
- In 1968, G. Veneziano proposed a function for describing the interaction of four particles.
- The generalization of the Veneziano amplitude to case of interaction of N particles:

$$A^{(N)}(\underline{\boldsymbol{k}}) = \int_{\mathbb{R}^{N-3}} \prod_{i=2}^{N-2} |x_i|_{\mathbb{R}}^{\boldsymbol{k}_1 \boldsymbol{k}_i} |1-x_i|_{\mathbb{R}}^{\boldsymbol{k}_{N-1} \boldsymbol{k}_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{\mathbb{R}}^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i.$$

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p-adic String Amplitudes

- The p-adic string theory started around 1987 with the work of three groups: Volovich, Freund and Witten, and Frampton and Okada.
- Freund and Olson noted that the integral expression for the Veneziano amplitude of the open string can be generalized to a p-adic integral.

- The p-adic string amplitudes for N particles have the form:

$$\boldsymbol{A}_p^{(N)}(\underline{\boldsymbol{k}}) := \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{\boldsymbol{k}_1 \boldsymbol{k}_i} |1 - x_i|_p^{\boldsymbol{k}_{N-1} \boldsymbol{k}_i} \prod_{2 \le i < j \le N-2} |x_i - x_j|_p^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i, \tag{1}$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalize Haar measure of $\mathbb{Q}_p^{N-3}$, $\underline{\boldsymbol{k}} = (\boldsymbol{k}_1, \dots, \boldsymbol{k}_N)$, $\boldsymbol{k}_i = (k_{0,i}, \dots, k_{25,i})$, $i = 1, \dots, N$, $N \ge 4$, (with Minkowski product $\boldsymbol{k}_i \boldsymbol{k}_j = -k_{0,i} k_{0,j} + k_{1,i} k_{1,j} + \cdots + k_{25,i} k_{25,j}$) satisfying

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0}, \ \boldsymbol{k}_i \boldsymbol{k}_i = 2 \text{ for } i = 1, \dots, N-1.$$

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$$Z_{\phi}(\underline{\boldsymbol{k}}) = \int_{\mathbb{Q}_p^{N-3}} \phi(x) \prod_{i=2}^{N-2} |x_i|_p^{\boldsymbol{k}_1 \boldsymbol{k}_i} |1 - x_i|_p^{\boldsymbol{k}_{N-1} \boldsymbol{k}_i} \prod_{2 \le i < j \le N-2} |x_i - x_j|_p^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i .$$

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Consider the following integral:

$$Z(s) = \int_{\mathbb{Q}_p} |x|_p^s \, dx.$$

Assume that this integral exists for some $s_0 \in \mathbb{R}$. Then

$$J_0(s_0) = \int_{\mathbb{Z}_p} |x|_p^{s_0} \, dx \text{ and } J_1(s_0) = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |x|_p^{s_0} \, dx$$

exist. But

$$J_0(s_0) = \frac{1 - p^{-1}}{1 - p^{-1-s_0}}.$$

converges for $s_0 > -1$,

$$J_1(s_0) = \sum_{j=1}^{\infty} \int_{p^{-j}\mathbb{Z}_p^{\times}} |x|_p^{s_0} \, dx = \sum_{j=1}^{\infty} p^{j(1+s_0)} < \infty$$

converges for $s_0 < -1$.

Regularization of p-adic String Amplitudes

Let $N \geq 4$ and $s_{ij} \in \mathbb{C}$ with $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N-1$. We define the *p-adic open string N-point zeta function* as

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} \, |1-x_i|_p^{s_{(N-1)i}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i, \tag{2}$$

where $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, $D = (N-4)(N-3)/2 + 2(N-3)$.

We use an approach inspired in the calculations presented in

Brekke Lee, Freund, Peter G. O., Olson Mark, Witten Edward, *Non-Archimedean string dynamics, Nuclear Phys. B 302 (1988), no. 3, 365-402.*

and in the Igusa's p-adic stationary phase formula.

We define for $I \subseteq T = \{2, ..., N-2\}$, *the sector attached to* I as

$$Sect(I) = \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; |x_i|_p \leq 1 \ \Leftrightarrow i \in I \right\}$$

$$\Rightarrow \mathbb{Q}_p^{N-3} = \sqcup_{I \subseteq T} Sect(I)$$

and

$$Z^{(N)}(\underline{s}; I) := \int_{Sect(I)} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i .$$

Then

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Then

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) = \sum_{I \subseteq T} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I).$$

$$Z^{(N)}\left(\underline{s};I\right)=Z^{(N)}\left(\underline{s};I,0\right)Z^{(N)}\left(\underline{s};I,1\right).$$

$$Z^{(N)}\left(\underline{s};I,0\right)=\int_{\mathbb{Z}_p^{|I|}}\prod_{i\in I}|x_i|_p^{s_{1i}}\,|1-x_i|_p^{s_{(N-1)i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I}dx_i$$

$$Z^{(N)}\left(\underline{s};I,1\right)=p^{M(\underline{s})}\int_{\mathbb{Z}_p^{|T\setminus I|}}\frac{\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in T\setminus I}}|y_i-y_j|_p^{s_{ij}}}{\prod_{i\in T\setminus I}|y_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum\limits_{2\leq j\leq N-2,j\neq i}s_{ij}}}\prod_{i\in T\setminus I}dx_i$$

where

$$M(\underline{s})=|T\setminus I|_p+\sum_{i\in T\setminus I}\left(s_{1i}+s_{(N-1)i}\right)+\sum_{\substack{2\leq i<j\leq N-2\\ i\in T\setminus I,j\in T}}s_{ij}+\sum_{\substack{2\leq i<j\leq N-2\\ i\in I,j\in T\setminus I}}s_{ij}.$$

- **Problem**: $Z^{(N)}(\underline{s})$ converges in some domain of $\mathbb{C}^D$.
- We showed that the functions $Z^{(N)}(\underline{s}; I, 0)$ and $Z^{(N)}(\underline{s}; I, 1)$:

  - Meromorphic continuations to the whole $\mathbb{C}^D$ as rational functions in the variables $p^{-s_{ij}}$.

  - Holomorphic on certain domain.

  - The intersection of all these domains contains an open and connected subset of $\mathbb{C}^D$.

$\Rightarrow$ $Z^{(N)}(\underline{s})$ has a meromorphic continuation to the whole $\mathbb{C}^D$ as a rational function in the variables $p^{-s_{ij}}$.

About the method...

- The algorithms compute recursively the integrals.
- These results are still valid if we replace $\mathbb{Q}_p$ by any non-Archimedean local field of arbitrary characteristic.

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Conditions of convergence

(C1) $|J| + \sum_{i \in J} (Re(s_{1i}) + Re(s_{(N-1)i})) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J}} Re(s_{ij}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus J, j \in J}} Re(s_{ij}) < 0$

(C2) $|L| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, i \in L}} Re(s_{ij}) > 0$

(C3) $1 + Re(s_{ij}) > 0$

(C4) $|M| + \sum_{i \in M} Re(s_{ti}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in M}} Re(s_{ij}) > 0, \ t \in \{1, N-1\}$

where $J, L, M \subseteq T$, $i, j \in T \cup \{1, ..., N-1\}$.

Main Results [Bocardo-Gaspar, García-Compeán, Zúñiga-Galindo]

Theorem (1), Bocardo-Gaspar, García-Compeán, Zúñiga-Galindo)

(1) The p-adic open string N-point zeta function, $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$, *gives rise to a holomorphic function on* $H(\mathbb{C})$, *which contains an open and connected subset of* $\mathbb{C}^D$. *Furthermore,* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ *admits an analytic continuation to* $\mathbb{C}^D$, *denoted also as* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$, *as a rational function in the variables* $p^{-s_{ij}}, i,j \in \{1, ..., N-1\}$. *The real parts of the poles of* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ *belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C4 with the symbols '$<$', '$>$' replaced by '$=$'. (2) If* $\underline{\boldsymbol{s}} = (s_{ij})_{ij} \in \mathbb{C}^D$, *with* $Re(s_{ij}) \geq 0$ $i,j \in \{1, \ldots, N-1\}$, *then* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) = +\infty$.

Regularization of p-adic String Amplitudes

- **Regularization:**

  We take the p-adic N-point zeta function $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ as regularizations of the amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ and define

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})\,|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1, N-1\}\,,\, j \in T \text{ or } i, j \in T.$$

  By Theorem 1), $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ are well-defined rational functions of the variables $p^{-s_{ij}}$, $i, j \in \{1, \ldots, N-1\}$, which agree with the original p-adic amplitudes when they converge.

Extending the results to finite extensions of $\mathbb{Q}_p$

Let $\mathbb{K}_e$ the unique un-ramified extension of $\mathbb{Q}_p$ of degree e, $N \geq 4$ and $s_{ij} \in \mathbb{C}$ with $s_{ij} = s_{ji}$, $1 \leq i < j \leq N-1$. We define *the open string N-point zeta function over* $\mathbb{K}_e$ as:

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e) = \int_{\mathbb{K}_e^{N-3}} \prod_{i=2}^{N-2} |x_i|_{\mathbb{K}_e}^{s_{1i}} |1-x_i|_{\mathbb{K}_e}^{s_{(N-1)i}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_{\mathbb{K}_e}^{s_{ij}} \prod_{i=2}^{N-2} dx_i,$$

where $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, $\prod_{i=2}^{N-2} dx_i$ is the Haar measure normalized of $\mathbb{K}_e^{N-3}$, $D = (N-4)(N-3)/2 + 2(N-3)$.

$$Z^{(N)}\left(\underline{s},\mathbb{K}_e\right)=\sum_{I\subseteq T} Z^{(N)}\left(\underline{s};I,0,\mathbb{K}_e\right)Z^{(N)}\left(\underline{s};I,1,\mathbb{K}_e\right),$$

$$Z^{(N)}\left(\underline{s};I,0,\mathbb{K}_e\right)=\int_{R_{\mathbb{K}_e}^{|I|}}\prod_{i\in I}|x_i|_{\mathbb{K}_e}^{s_{1i}}\,|1-x_i|_{\mathbb{K}_e}^{s_{(N-1)i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_{\mathbb{K}_e}^{s_{ij}}\prod_{i\in I}dx_i$$

$$Z^{(N)}\left(\underline{s};I,1,\mathbb{K}_e\right)=p^{eM(\underline{s})}\int_{R_{\mathbb{K}_e}^{|T\setminus I|}}\frac{\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in T\setminus I}}|y_i-y_j|_{\mathbb{K}_e}^{s_{ij}}}{\prod_{i\in T\setminus I}|y_i|_{\mathbb{K}_e}^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2,j\neq i}s_{ij}}}\prod_{i\in T\setminus I}dx_i$$

where

$$M(\underline{s})=|T\setminus I|_{\mathbb{K}_e}+\sum_{i\in T\setminus I}\left(s_{1i}+s_{(N-1)i}\right)+\sum_{\substack{2\leq i<j\leq N-2\\ i\in T\setminus I,j\in T}}s_{ij}+\sum_{\substack{2\leq i<j\leq N-2\\ i\in I,j\in T\setminus I}}s_{ij}.$$

Conditions of convergence

(C1) $|J| + \sum_{i\in J}(Re(s_{1i}) + Re(s_{(N-1)i})) + \sum_{\substack{2\le i<j\le N-2\\ i\in J}} Re(s_{ij}) + \sum_{\substack{2\le i<j\le N-2\\ i\in T\setminus J, j\in J}} Re(s_{ij}) < 0$

(C2) $|L| - 1 + \sum_{\substack{2\le i<j\le N-2\\ i,i\in L}} Re(s_{ij}) > 0$

(C3) $1 + Re(s_{ij}) > 0$

(C4) $|M| + \sum_{i\in M} Re(s_{ti}) + \sum_{\substack{2\le i<j\le N-2\\ i,j\in M}} Re(s_{ij}) > 0,\ t \in \{1, N-1\}$

where $J, L,\ M \subseteq T,\ i, j \in T \bigcup \{1, N-1\}$.

Theorem (Bocardo-Gaspar, García-Compeán, Zúñiga-Galindo)

The open string N-point zeta function $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$*, gives rise to a holomorphic function on* $H(\mathbb{C})$*, which contains an open and connected subset of* $\mathbb{C}^D$*. Furthermore,* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, K_e)$ *admits an analytic continuation to* $\mathbb{C}^D$*, denoted also as* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$*, as a rational function in the variables* $p^{-es_{ij}}, i, j \in \{1, \ldots, N-1\}$*. The real parts of the poles of* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ *belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C4 with the symbols '$<$', '$>$' replaced by '$=$'. (2) If* $\underline{\boldsymbol{s}} = (s_{ij})_{ij} \in \mathbb{C}^D$*, with* $Re(s_{ij}) \geq 0 \; i, j \in \{1, \ldots, N-1\}$*, then* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e) = +\infty$*.*

String amplitudes over $\mathbb{K}_e$

- The open string N-point tree amplitudes over $\mathbb{K}_e$:

$$\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}}, \mathbb{K}_e\right) := \int_{\mathbb{K}_e^{N-3}} \prod_{i=2}^{N-2} |x_i|_{\mathbb{K}_e}^{\boldsymbol{k}_1 \boldsymbol{k}_i} |1 - x_i|_{\mathbb{K}_e}^{\boldsymbol{k}_{N-1} \boldsymbol{k}_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{\mathbb{K}_e}^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i,$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{K}_e^{N-3}$, $\underline{\boldsymbol{k}} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, $\boldsymbol{k}_i = (k_{0,i}, \ldots, k_{25,i})$, $i = 1, \ldots, N$, $N \geq 4$, (with Minkowski product $\boldsymbol{k}_i \boldsymbol{k}_j = -k_{0,i} k_{0,j} + k_{1,i} k_{1,j} + \cdots + k_{25,i} k_{25,j}$) such that

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0}, \ \boldsymbol{k}_i \boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N-1.$$

Regularization of $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}};\mathbb{K}_e)$

We take the open string N-point topological zeta function $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}},\mathbb{K}_e)$ as regularizations of the amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}},\mathbb{K}_e)$. More precisely, we define

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}};\mathbb{K}_e) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}},\mathbb{K}_e)\mid_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i\in\{1,\dots,N-1\},\ j\in T \text{ or } i,j\in T,$$

where $T=\{2,\dots,N-2\}$.
$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}},\mathbb{K}_e)$ are well-defined rational functions of the variables $p^{-\boldsymbol{k}_i\boldsymbol{k}_j}$, i, $j\in\{1,\dots,N-1\}$. which agree with integrals (1) when they converge.

String topological zeta functions

- For the multivariate local zeta functions $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e\right)$ and $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e\right)$, we can apply the theory of Denef-Loeser, to define

$$\boldsymbol{Z}_{0,top}^{(N)}\left(\underline{\boldsymbol{s}}; I\right) = \lim_{e \to 0} \boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e\right) \quad \text{and}$$
$$\boldsymbol{Z}_{1,top}^{(N)}\left(\underline{\boldsymbol{s}}; T \smallsetminus I\right) = \lim_{e \to 0} \boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e\right),$$

which are elements of $\mathbb{Q}\left(s_{ij}, i, j \in \{1, \ldots, N-1\}\right)$, the field of rational functions in the variables s_{ij}, with coefficients in $\mathbb{Q}$.

Denef and F. Loeser, Caractéristiques d'Euler Poincaré, fonctions zêta locales et modifications analytiques J. Amer. Math. Soc. 5 (1992), no. 4, 705720.

- The limit $e \to 0$ makes sense because one can l-adically interpolate $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, i, \mathbb{K}_e)$ as a function of e. This means that there exists $\kappa \in \mathbb{N} \setminus \{0\}$ and a meromorphic function in the variables $\boldsymbol{s}$ and e, $\boldsymbol{Z}_l(\boldsymbol{s}; I, i, e)$ on $\mathbb{Z}_l^D \times (\kappa\mathbb{Z}_l)$ such that holds

$$\boldsymbol{Z}_l(\boldsymbol{s}; I, i, e)|_{\mathbb{Z}^D \times \kappa\mathbb{N}} = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, i, \mathbb{K}_e)|_{\mathbb{Z}^D \times \kappa\mathbb{N}}.$$

for each $i = 0, 1$.

- The limits of the multivariate local zeta functions being l-adic.
- We define the open string N-point topological zeta function as

$$Z_{top}^{(N)}(\underline{s}) := \sum_{I \subseteq T} Z_{0,top}^{(N)}(\underline{s}; I)\, Z_{1,top}^{(N)}(\underline{s}; I)$$

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$$\boldsymbol{Z}_l(\boldsymbol{s}; I, i, e)\,|_{\mathbb{Z}^D \times \kappa\mathbb{N}} = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, i, \mathbb{K}_e)\,|_{\mathbb{Z}^D \times \kappa\mathbb{N}}.$$

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- The limits of the multivariate local zeta functions being l-adic.
- We define the open string N-point topological zeta function as

$$\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}) := \sum_{I \subseteq T} \boldsymbol{Z}_{0,top}^{(N)}(\underline{\boldsymbol{s}}; I)\, \boldsymbol{Z}_{1,top}^{(N)}(\underline{\boldsymbol{s}}; I)$$

Theorem

The open string N-point topological zeta function $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})$ *is a rational function from* $\mathbb{Q}(s_{ij}, i, j \in \{1, \ldots, N-1\})$*. The real parts of the possible poles of* $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})$ *belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C4 with the symbols '*$<$*', '*$>$*' replaced by '*$=$*'. (2) If* $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$*, with* $\mathrm{Re}(s_{ij}) \geq 0$ *for* $i, j \in \{1, \ldots, N-1\}$*, then* $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}) = +\infty$*.*

- We define the string amplitude underlying the topological zeta function $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})$ as

$$\boldsymbol{A}_{top}^{(N)}(\boldsymbol{k}) = \boldsymbol{Z}_{top}^{(N)}(\boldsymbol{s})|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j}$$

with $i \in \{1, .., N-1\}$, $j \in T$ or $i, j \in T$ where $T = \{e, ..., N-2\}$, which are rational functions of the variables $k_i k_j$, $i, j \in \{1, ...N-1\}$.

Part II.