# Non-Archimedean Reaction-Ultradiffusion Equations and Complex Hierarchic Systems

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# Presentation Plan

- Notation
- The $p$-adic heat equation
- $p$-adic models of complex systems
- $p$-adic Reaction-ultradiffusion equations:

W. A. Zúñiga-Galindo, Non-Archimedean Reaction-Ultradiffusion Equations and Complex Hierarchic Systems, arXiv:1604.06471.

# Abstract

- We initiate the study of non-Archimedean reaction-ultradiffusion equations and their connections with models of complex hierarchic systems.
- From a mathematical perspective, the equations studied here are the $p$-adic counterpart of the integro-differential models for phase separation introduced by Bates and Chmaj.
- Our equations are also generalizations of the ultradiffusion equations on trees studied in the 80's by Ogielski, Stein, Bachas, Huberman, among others, and also generalizations of the master equations of the Avetisov et al. models, which describe certain complex hierarchic systems.
- From a physical perspective, our equations are gradient flows of non-Archimedean free energy functionals and their solutions describe the macroscopic density profile of a bistable material whose space of states has an ultrametric structure.

# The field of p-adic numbers

The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order of $x$*. We extend the $p-$adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a complete ultrametric space.

- For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n; ||x - a||_p \leq p^r\}$ *the ball of radius* $p^r$ *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$.

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- We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n; ||x - a||_p = p^r\}$ *the sphere of radius* $p^r$ *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$.

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- The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined by

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p} \chi_p(-\xi \cdot x)\varphi(\xi)\, d^n x,\ \xi \in \mathbb{Q}_p^n,$$

where $\chi_p(\cdot)$ is the standard additive character of $\mathbb{Q}_p$, $\xi \cdot x = \sum_i \xi_i x_i$ and $d^n x$ is the normalized Haar measure on $\mathbb{Q}_p^n$.

# The p-adic heat equation

- $$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (D^{\alpha} u)(x,t) = h(x,t), & x \in \mathbb{Q}_p,\ t > 0 \\ u(x,0) = \varphi(x), \end{cases}$$

  where
  $$(D^{\alpha}\varphi)(x) := \mathcal{F}^{-1}_{\xi \to x}\left(|\xi|_p^{\alpha}\, \mathcal{F}_{x \to \xi}\varphi\right), \alpha > 0,$$
  is the **Vladimirov operator.**

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- $$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \left(\frac{d^2 u}{dx^2}\right)(x,t) = f\left(x,t\right), \quad x \in \mathbb{R},\ t > 0 \\ u\left(x,0\right) = \varphi\left(x\right). \end{cases}$$

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- The heat kernel:

$$Z(x,t) = \mathcal{F}^{-1}_{\xi \to x}\left(e^{-t|\xi|_p^{\alpha}}\right) = \int_{\mathbb{Q}_p} \chi_p(x\xi)\, e^{-t|\xi|_p^{\alpha}} d\xi.$$

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- 

$$\begin{aligned} u(x,t) &= \int_{\mathbb{Q}_p} Z(x-\xi,t)\,\varphi(\xi)\,d\xi + \\ & \int_0^t\int_{\mathbb{Q}_p} Z(x-\xi,t-\tau)\,h(\xi,\tau)\,d\xi d\tau. \end{aligned}$$

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- $\lim_{t\to 0^+} Z_t(x) = \delta(x)$ en $\mathcal{D}'$
- $Z_t * Z_{t'} = Z_{t+t'},\ t, t' > 0$
- $p(t,x,y) = Z(x-y,t)$ is a probability density ( space and time homogeneous)

# The p-adic heat equation

- $P(t,x,B) = \int_B p(t,x,y)\,dy$ expresses the probability that a particle which has started out from the point $x$ is in the set $B$ at the time $t$.

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- $Z(x,t)$ is the transition density of a bounded right-continuous Markov process without second kind discontinuities.

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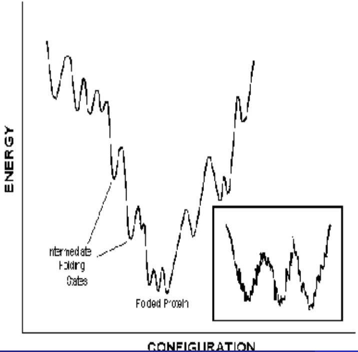

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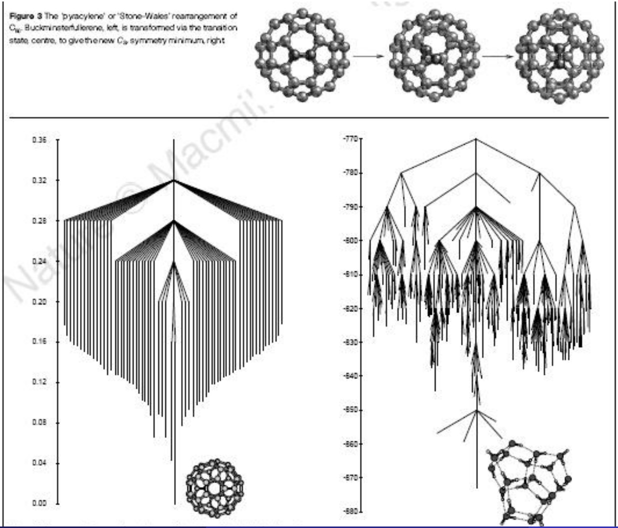
- The idea is to study the **kinetics** generated by transitions between groups of states (basins).
- Minimal basins correspond to local minima of energy.
- **A complex landscape is approximated by a disconnectivity graph (an ultrametric space) and the distribution function of activation energies.**

# The energy landscape

D. J. Wales, M. A. Miller and T. R. Walsh,Archetypal energy landscapes, Nature, 394 758-760 (1998)



## Interbasin kinetics

- The transitions between basins are described by the following equations:

$$\frac{\partial f(i,t)}{\partial t} = \sum_j T(j,i)\, f(j,t)\, v(j) - \sum_j T(i,j)\, f(i,t)\, v(i),$$

where the indices $i$,$j$ number the states of the system (which correspond to local minima of energy), $T(i,j) \geq 0$ is the probability per unit time of a transition from $i$ to $j$, and the $v(j) > 0$ are the basin volumes.

# Interbasin kinetics

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  where the indices $i$,$j$ number the states of the system (which correspond to local minima of energy), $T\left(i,j\right)\geq 0$ is the probability per unit time of a transition from $i$ to $j$, and the $v(j)>0$ are the basin volumes.

- From a physical point of view, the above equation **must be** a diffusion equation on a tree.

# Interbasin kinetics

- Avetisov, V. A.; Bikulov, A. H.; Kozyrev, S. V.; Osipov, V. A. p-adic models of ultrametric diffusion constrained by hierarchical energy landscapes. J. Phys. A 35 (2002), no. 2, 177–189.

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- Take $v(j) = 1$ and $T(i,j) = q\left(|i-j|_p\right)$, then the master equation takes the form

$$\frac{\partial f(i,t)}{\partial t} = \int\limits_{p^M\mathbb{Z}_p/p^N\mathbb{Z}_p} q\left(|i-j|_p\right)\left(f(i,t) - f(j,t)\right) d\mu(j), \ M < N,$$

where the integration (summation!) is with respect to the Haar measure on the discrete group $p^M\mathbb{Z}_p/p^N\mathbb{Z}_p$. By taking the formal limits $M \to -\infty$ and $N \to +\infty$ we get a $p$-adic diffusion equation.

# Ultrametricity in physics

- The general $p$-adic master equation describing a Markovian process of a random walk in $\mathbb{Q}_p$ can be written as

$$\frac{\partial f\left(x,t\right)}{\partial t}=\int_{\mathbb{Q}_p}\left[w\left(x\left|y\right.\right)f\left(y,t\right)-w\left(y\left|x\right.\right)f\left(x,t\right)\right]dy,$$

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$x\in\mathbb{Q}_p, t\geq 0.$

- The function $f(x,t):\mathbb{Q}_p\times\mathbb{R}_+\rightarrow\mathbb{R}_+$ is a probability density distribution, so that $\int_B f\left(x,t\right)dx$ is the probability of finding the system in a domain $B\subset\mathbb{Q}_p$ at the instant $t$. The function $w\left(x\left|y\right.\right):\mathbb{Q}_p\times\mathbb{Q}_p\rightarrow\mathbb{R}_+$ is the probability of the transition from state $y$ to state $x$ per unit of time.

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- The transition from state $y$ to a state $x$ can be visualized as overcoming the energy barrier separating these states.

# Motivations

- $$\begin{aligned} \frac{dP}{dt} &= AP,\ A = [A_{i,j}] = \left[A\left(\left|i-j\right|_p\right)\right],\ i,j \in \mathbb{Z}_p/p^L\mathbb{Z}_p \\ &\rightarrow p-\text{adic heat equation as } L \rightarrow \infty \end{aligned}$$

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- To study non-linear (physically relevant ) $p$-adic equations related with $p-$adic heat equations.

# P-adic Reaction-Ultradiffusion Equations

The $p$-adic limit of master equations have the form:

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\left[u(y,t) - u(x,t)\right] d^n y, \tag{1}$$

$x \in \mathbb{Q}_p^n$, $t \geq 0$. The function $u(x,t) : \mathbb{Q}_p^n \times \mathbb{R}_+ \to \mathbb{R}_+$ is a probability density distribution, so that $\int_B u(x,t)\, d^n x$ is the probability of finding the system in a domain $B \subset \mathbb{Q}_p^n$ at the instant $t$. The function $J\left(\|x-y\|_p\right) : \mathbb{Q}_p^n \times \mathbb{Q}_p^n \to \mathbb{R}_+$ is the probability of the transition from state $y$ to state $x$ per unit of time. It is known that for many $J$'s, equations of type (1) are ultradiffusion equations i.e. they are $p$-adic counterparts of the classical heat equations. More precisely, the fundamental solution of (1) is the transition density of a bounded right-continuous Markov process without second kind discontinuities.

# P-adic Reaction-Ultradiffusion Equations

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\left[u(y,t)-u(x,t)\right] d^n y, \;\; J \in L^1\left(\mathbb{Q}_p^n\right).$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{1-p^{\alpha}}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \frac{\left[u(y,t)-u(x,t)\right]}{\|x-y\|_p^{\alpha+n}} d^n y,$$

$$\frac{1-p^{\alpha}}{1-p^{-\alpha-n}} \frac{1}{\|x\|_p^{\alpha+n}} \notin L^1\left(\mathbb{Q}_p^n\right).$$

- These two equations have the same physical meaning, but mathematically speaking, they are different objects.

# P-adic Reaction-Ultradiffusion Equations

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)[u(y,t)-u(x,t)]\,d^n y,\ \ J \in L^1\left(\mathbb{Q}_p^n\right).$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{1-p^{\alpha}}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \frac{[u(y,t)-u(x,t)]}{\|x-y\|_p^{\alpha+n}} d^n y,$$

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- Anselmo Torresblanca-Badillo, W. A. Zúñiga-Galindo , Ultrametric Diffusion, Exponential Landscapes, and the First Passage Time Problem.arXiv:1511.08757

# P-adic Reaction-Ultradiffusion Equations

- W. A. Zúñiga-Galindo, Non-Archimedean Reaction-Ultradiffusion Equations and Complex Hierarchic Systems, arXiv:1604.06471.

# P-adic Reaction-Ultradiffusion Equations

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- We study equations of type

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\left[u(y,t)-u(x,t)\right] d^n y - \lambda f\left(u(x,t)\right), \tag{2}$$

where $J\left(\|x\|_p\right) \geq 0$, $\int_{\mathbb{Q}_p^n} J\left(\|x\|_p\right) d^n x = 1$, $\lambda > 0$ sufficiently large and $f$ is (for instance) a polynomial having roots in $-1$, $0$, $1$.

# P-adic Reaction-Ultradiffusion Equations

- Formally, equation (2) is the $L^2$-gradient flow of the following non-Archimedean Helmholtz free-energy functional:

$$E[\varphi] = \frac{1}{4}\int_{\mathbb{Q}_p^n}\int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\{\varphi(x)-\varphi(y)\}^2 d^n x d^n y \quad (3)$$
$$+\lambda\int_{\mathbb{Q}_p^n} W(\varphi(x))\, d^n x,$$

where $\varphi$ is a function taking values in the interval $[-1,1]$ and $W$ is a double-well potential.

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where $\varphi$ is a function taking values in the interval $[-1,1]$ and $W$ is a double-well potential.

- The scalar function $\varphi$ represents the macroscopic density profile of a system which has two equilibrium pure phases described by $\varphi \equiv 1$ and $\varphi \equiv -1$. The integral $\int_{\mathbb{Q}_p^n} W(\varphi(x))\, d^n x$ in the right side of (3) forces the minimizer of $E$ to take values close to $+1$ and $-1$ (phase separation) while the double integral represents an interaction energy integral which penalizes the spatial inhomogenety of the system.

# P-adic Reaction-Ultradiffusion Equations

$$\lim_{\epsilon \to 0} \frac{E[\varphi + \epsilon\theta] - E[\varphi]}{\epsilon} =$$

$$\left( \int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)[\varphi(y) - \varphi(x)]\, d^n y, \theta(x) \right)$$

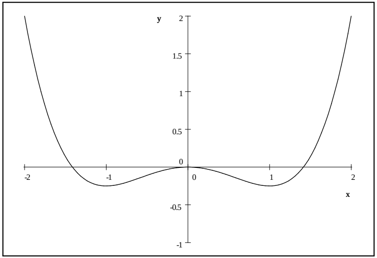
$$+\lambda \left( \int_{\mathbb{Q}_p^n} W(\varphi(x))\, \theta(x)\, d^n x \right)$$

$$\overset{formally}{=} \int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)[\varphi(y) - \varphi(x)]\, d^n y - \lambda f(\varphi(x))$$

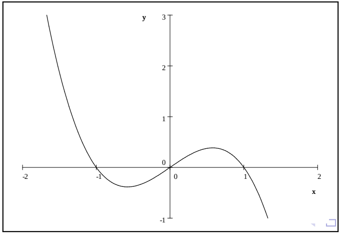
$$= -A\varphi(x) - \lambda f(\varphi(x)) = -\nabla\varphi(x)$$

# P-adic Reaction-Ultradiffusion Equations

$W(u) = \frac{u^2}{4}(u^2 - 2)$




$f(u) = -u(u^2 - 1) = \dot{W}(u)$



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# P-adic Reaction-Ultradiffusion Equations

- Equations of (2) can be well-approximated in finite dimensional real spaces by ODE's.
- In a suitable basis, where the unknown function is identified with the column vector $[u(\mathbf{i},t)]_{\mathbf{i}\in G_N^n}$, these equations have the form

$$\frac{\partial}{\partial t}[u(\mathbf{i},t)]_{\mathbf{i}\in G_N^n} = -A^{(N)}[u(\mathbf{i},t)]_{\mathbf{i}\in G_N^n} - \lambda\left[f(u(\mathbf{i},t))\right]_{\mathbf{i}\in G_N^n}, \quad (4)$$

  where $A^{(N)}$ is the matrix representation of a linear operator that approximates, in a suitable finite dimensional vector space, the integral operator involving the function $J$ in the right-side of (2). Equation (4) is $L^2$-gradient flow of a 'finite' Helmholtz energy functional.

# Some Functional Spaces and Operators

- We define $X_\infty(\mathbb{Q}_p^n) := X_\infty = \overline{\left(\mathcal{D}(\mathbb{Q}_p^n), \|\cdot\|_\infty\right)}$, where $\|\phi\|_\infty = \sup_{x\in\mathbb{Q}_p^n} |\phi(x)|$ and the bar means the completion with respect the metric induced by $\|\cdot\|_\infty$. We also use $\|\cdot\|_\infty$ to denote the extension of $\|\cdot\|_\infty$ to $X_\infty$. Notice that all the functions in $X_\infty$ are continuous and that

$$X_\infty \subset C_0 :=$$

$$\left(\left\{ f : \mathbb{Q}_p^n \to \mathbb{R}; f \text{ continuous with } \lim_{\|x\|_p\to\infty} f(x) = 0 \right\}, \|\cdot\|_\infty \right).$$

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$$X_\infty \subset C_0 :=$$

$$\left(\left\{f : \mathbb{Q}_p^n \to \mathbb{R}; f \text{ continuous with } \lim_{\|x\|_p\to\infty} f(x) = 0\right\}, \|\cdot\|_\infty\right).$$

- On the other hand, since $\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $C_0$, we conclude that $X_\infty = C_0$. In a more general case, if $K$ is an open subset of $\mathbb{Q}_p^n$, we define $X_\infty(K) = \overline{\left(\mathcal{D}(K), \|\cdot\|_\infty\right)}$.

# Some Functional Spaces and Operators

- We set
$$X_N := \left(\mathcal{D}_N^{-N}\left(\mathbb{Q}_p^n\right), \|\cdot\|_\infty\right) \text{ for } N \geq 1.$$
Any $\varphi \in X_N$ has support in $B_N^n = \left(p^{-N}\mathbb{Z}_p\right)^n$, and $\varphi$ satisfies $\varphi(x + x') = \varphi(x)$ for $x' \in B_{-N}^n = \left(p^N\mathbb{Z}_p\right)^n$. In addition, $B_{\pm N}^n$ are additive subgroups and $G_N^n := B_N^n / B_{-N}^n$ is a finite group with $\#G_N^n := p^{2Nn}$ elements.

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$$\mathbf{i}_j = a_{-N}^j p^{-N} + a_{-N+1}^j p^{-N+1} + \ldots + a_0^j + a_1^j p + \ldots + a_{N-1}^j p^{N-1} \tag{5}$$
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- Any $\varphi \in X_N$ can be represented as $\varphi(x) = \sum_{\mathbf{i} \in G_N^n} \varphi(\mathbf{i})\, \Omega\left(p^N \|x - \mathbf{i}\|_p\right)$, with $\varphi_{\mathbf{i}} \in \mathbb{R}$, where $\Omega\left(p^N \|x - x_0\|_p\right)$ denotes the characteristic function of the ball $x_0 + \left(p^N\mathbb{Z}_p\right)^n$.

# Some Functional Spaces and Operators

- $\left\{\Omega\left(p^N \|x-\mathbf{i}\|_p\right)\right\}_{\mathbf{i}\in G_N^n}$ is a basis of $\mathcal{D}_N^{-N}$.

**Lemma**

$\lim_{N\to\infty}\|\varphi - P_N\varphi\|_\infty = 0$ *for any* $\varphi \in X_\infty$.

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- Then $\|\varphi\|_\infty = \max_{\mathbf{i}} |\varphi_{\mathbf{i}}|$. Hence $X_N$ is isomorphic as a Banach space to $\left(\mathbb{R}^{\#G_N^n}, \|\cdot\|_{\mathbb{R}}\right)$, where $\left\|\left(t_1,\ldots,t_{\#G_N^n}\right)\right\|_{\mathbb{R}} = \max_{1\leq j\leq \#G_N^n} |t_j|$.

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- We now define for $N\geq 1$, $P_N : X_\infty \to X_N$ as $P_N\varphi(x) = \sum_{\mathbf{i}\in G_N^n} \varphi(\mathbf{i})\,\Omega\left(p^N \|x-\mathbf{i}\|_p\right)$.

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- We now define for $N\geq 1$, $P_N : X_\infty \to X_N$ as $P_N\varphi(x) = \sum_{\mathbf{i}\in G_N^n} \varphi(\mathbf{i})\,\Omega\left(p^N \|x-\mathbf{i}\|_p\right)$.
- Therefore $P_N$ is a linear bounded operator, indeed, $\|P_N\| \leq 1$.

**Lemma**

$\lim_{N\to\infty} \|\varphi - P_N\varphi\|_\infty = 0$ *for any* $\varphi \in X_\infty$.

- We denote by $E_N$ , $N \geq 1$, the embedding $X_N \rightarrow X_\infty$. The following result is a consequence of the above observations. If $Z$, $Y$ are real Banach spaces, we denote by $\mathfrak{B}(Z, Y)$, the space of all linear bounded operators from $Z$ into $Y$.

**Lemma (Condition A)**

*With the above notation, the following assertions hold:*
*(i)* $X_\infty$, $X_N$ *for* $N \geq 1$, *are real Banach spaces, all with the norm* $\|\cdot\|_\infty$;
*(ii)* $P_N \in \mathfrak{B}(X_\infty, X_N)$ *and* $\|P_N\varphi\|_\infty \leq \|\varphi\|_\infty$ *for any* $N \geq 1$, $\varphi \in X_\infty$;
*(iii)* $E_N \in \mathfrak{B}(X_N, X_\infty)$ *and* $\|E_N\varphi\|_\infty = \|\varphi\|_\infty$ *for any* $N \geq 1$, $\varphi \in X_N$;
*(iv)* $P_N E_N \varphi = \varphi$ *for* $N \geq 1$, $\varphi \in X_N$.

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*(iv) $P_N E_N \varphi = \varphi$ for $N \geq 1$, $\varphi \in X_N$.*

- Set $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$. We fix a continuous function $J : \mathbb{R}_+ \to \mathbb{R}_+$, and take $J(x) = J(||x||_p)$ for $x \in \mathbb{Q}_p^n$, thus $J(x)$ is a *radial function* on $\mathbb{Q}_p^n$. We assume that $\int_{\mathbb{Q}_p^n} J(||x||_p) d^n x = 1$.

# The operators AN, A

## Lemma

*The following assertions hold:*
*(i) set* $J_N(||x||_p) := J(||x||_p)\Omega\left(p^{-N}\left\|x\right\|_p\right)$ *for* $N \geq 1$*. Then*

$$J_N(||x||_p) * P_N\varphi\left(x\right) = \Omega\left(p^{-N}\left\|x\right\|_p\right)\left\{J(||x||_p) * P_N\varphi\left(x\right)\right\}$$

*for* $\varphi\left(x\right) \in X_\infty$*;*
*(ii) define for* $N \geq 1$*,*

$$\begin{array}{llll} A_N: & X_N & \rightarrow & X_N \\ & \phi\left(x\right) & \rightarrow & -\int_{B_N^n} J_N(||x-y||_p)\left\{\phi\left(y\right)-\phi\left(x\right)\right\}d^ny. \end{array}$$

*Then* $A_N$ *is a well-defined linear bounded operator.*

We define

$$\begin{array}{rccl} A: & X_\infty & \to & X_\infty \\ & \varphi(x) & \to & A\varphi(x) = -\left\{ J\left(\|x\|_p\right) * \varphi(x) - \varphi(x) \right\}. \end{array} \tag{6}$$

**Remark**

*Notice that* $A\varphi(x) = -\int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\{\varphi(y)-\varphi(x)\}\, d^n y$ *since* $\int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right) d^n y = 1$.

**Lemma**

*The operator* $A : X_\infty \to X_\infty$ *is a linear and bounded. In addition, the spectrum of* $A$, $\sigma(A)$, *is contained in the interval* $[0, 2]$.

# The Matrix Representation of operators $A_N$ and Markov Chains

By using the basis $\left\{\Omega\left(p^N \|x-\mathbf{i}\|_p\right)\right\}_{\mathbf{i}\in G_N^n}$ we identify $X_N$ with $\left(\mathbb{R}^{\# G_N^n}, \|\cdot\|_{\mathbb{R}}\right)$, thus operator $A_N$ is given by a matrix. This matrix is computed by means of the following two lemmas.

**Lemma**

*Set* $\mathfrak{a}(x,\mathbf{i}) := J_N\left(\|x\|_p\right) * \Omega\left(p^N \|x-\mathbf{i}\|_p\right)$ *for* $x \in B_N^n$, $\mathbf{i} \in G_N^n$. *Let* $\widetilde{x}$ *denote the image of* $x$ *under the canonical map* $B_N^n \rightarrow G_N^n$. *Then*

$$\mathfrak{a}(x,\mathbf{i}) = \mathfrak{a}(\widetilde{x},\mathbf{i}) = \begin{cases} p^{-Nn} J\left(p^{-ord(\widetilde{x}-\mathbf{i})}\right) & \text{if} \quad ord(\widetilde{x}-\mathbf{i}) \neq +\infty \\ \int\limits_{(p^N\mathbb{Z}_p)^n} J\left(\|y\|_p\right) d^n y & \text{if} \quad ord(\widetilde{x}-\mathbf{i}) = +\infty. \end{cases}$$

# The Matrix Representation of operators AN and Markov Chains

## Lemma

*The matrix for operator* $A_N$ *acting on* $X_N$ *is* $A^{(N)} = \left[A_{\mathbf{ki}}^{(N)}\right]_{\mathbf{k},\mathbf{i}\in G_N^n} = [j_N \delta_{\mathbf{ki}} - \mathfrak{a}_{\mathbf{ki}}]_{\mathbf{k},\mathbf{i}\in G_N^n}$, *where* $\mathfrak{a}_{\mathbf{ki}} := \mathfrak{a}(\mathbf{k},\mathbf{i})$ *and* $\delta_{\mathbf{ki}}$ *denotes the Kronecker delta.*

## Lemma

$-A^{(N)}$ *is a Q-matrix, i.e.* $-A_{\mathbf{ij}}^{(N)} \geq 0$ *for* $\mathbf{i} \neq \mathbf{j}$ *with* $\mathbf{i}, \mathbf{j} \in G_N^n$, *and* $A_{\mathbf{ii}}^{(N)} = -\sum_{\mathbf{j}\neq\mathbf{i}} A_{\mathbf{ij}}^{(N)}$.

# The Matrix Representation of operators AN and Markov Chains

## Theorem

*(i) Set* $P^{(N)}(t) := e^{-tA^{(N)}}$, $t \geq 0$. *Then* $P^{(N)}(t)$ *is a semigroup of nonnegative matrices with* $P^{(N)}(0) = \mathbb{E}$, *the identity matrix, which satisfies*

$$\frac{\partial P^{(N)}(t)}{\partial t} + A^{(N)} P^{(N)}(t) = 0$$

*and* $P^{(N)}(t)\mathbf{1} = \mathbf{1}$ *for* $t \geq 0$.
*(ii) The function* $P^{(N)}(t-s)$, $t \geq s \geq 0$, *is the transition function of a homogeneous Markov chain with state space* $G_N^n$. *Furthermore, this stochastic process has right-continuous piece-wise-constant paths.*

# Non-Archimedean Helmholtz Free-Energy Functionals

- We define for $\varphi \in X_N$, $\lambda > 0$,

$$E_N(\varphi) = \frac{1}{4} \int_{B_N^n} \int_{B_N^n} J_N\left(\|x-y\|_p\right) \{\varphi(x) - \varphi(y)\}^2 d^n x d^n y + \lambda \int_{B_N^n} W(\varphi(x)) d^n x, \tag{7}$$

where $J_N\left(\|x\|_p\right)$ is as before, $\varphi$ is a scalar density function defined on $B_N^n$ that takes values in $[-1,1]$, $W:\mathbb{R}\rightarrow\mathbb{R}$, with derivative $f \in C^2(\mathbb{R})$, is a double-well potential having (not necessarily equal) minima at $\pm 1$.

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- The function $\varphi$, the order parameter, represents the macroscopic density profile of a system which has two equilibrium pure phases described by the profiles $\varphi \equiv 1$ and $\varphi \equiv -1$, and $-1 < \varphi < 1$ represents the 'interface'. The function $J_N$ is a positive, possibly anisotropic, interaction potential which vanishes at infinity.

# Non-Archimedean Helmholtz Free-Energy Functionals

## Lemma

*(i) By identifying* $\varphi(x)$ *with the vector* $[\varphi(\mathbf{i})]_{\mathbf{i}\in G_N^n}$, *i.e. by identifying* $X_N$ *with* $\mathbb{R}^{\#G_N^n}$, *we have*

$$E_N\left([\varphi(\mathbf{i})]_{\mathbf{i}\in G_N^n}\right) = \frac{j_N p^{-Nn}}{2}\sum_{\mathbf{i}\in G_N^n}\varphi^2(\mathbf{i}) - \frac{p^{-Nn}}{2}\sum_{\mathbf{i},\mathbf{j}\in G_N^n}\mathfrak{a}_{\mathbf{ij}}\varphi(\mathbf{i})\varphi(\mathbf{j}) + \lambda p^{-Nn}\sum_{\mathbf{i}\in G_N^n}W(\varphi(\mathbf{i})),$$

*where* $[\mathfrak{a}_{\mathbf{ij}}]_{\mathbf{i},\mathbf{j}\in G_N^n}$ *is the matrix defined in Lemma 5.*

### Lemma

*(ii) We assume that $\varphi$ depends on $\mathbf{i} \in G_N^n$ and $t \geq 0$. The gradient flow in the Euclidean space $\mathbb{R}^{\#G_N^n}$ of the functional $E_N : \mathbb{R}^{\#G_N^n} \rightarrow \mathbb{R}$ is the evolution in $\mathbb{R}^{\#G_N^n}$ given by*

$$\begin{aligned} \frac{\partial}{\partial t} \left[\varphi\left(\mathbf{i}, t\right)\right]_{\mathbf{i} \in G_N^n} &= -\nabla E_N \left( \left[\varphi\left(\mathbf{i}, t\right)\right]_{\mathbf{i} \in G_N^n} \right) \\ &= -p^{-Nn} A^{(N)} \left[\varphi\left(\mathbf{i}, t\right)\right]_{\mathbf{i} \in G_N^n} - \lambda p^{-Nn} \left[f\left(\varphi\left(\mathbf{i}, t\right)\right)\right]_{\mathbf{i} \in G_N^n}, \end{aligned} \tag{8}$$

*where $A^{(N)}$ is the matrix defined in Lemma 6.*

### Remark

*Notice that in $X_N$, (8) can be written as*

$$\frac{\partial}{\partial t} \varphi\left(x, t\right) = -A_N \varphi\left(x, t\right) - \lambda f\left(\varphi\left(x, t\right)\right). \tag{9}$$

# Non-Archimedean Helmholtz Free-Energy Functionals

- Consider $\left(G_N^n, \|\cdot\|_p\right)$ as a finite ultrametric space. Then (8) is reaction-ultradiffusion equation in $\left(G_N^n, \|\cdot\|_p\right)$, which is the $L^2$-gradient of an energy functional defined on $\left(G_N^n, \|\cdot\|_p\right)$.

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- We initiate the study of these equations and their 'limits' as $N$ tends to infinity. In the special case $f \equiv 0$, by a physical argument involving the parametrization of Parisi matrices by $p$-adic numbers, Avetisov et al. showed that the 'limit' of an equation of type (8) as $N$ tends to infinity is

$$\frac{\partial}{\partial t}\varphi\left(x,t\right)=-A\varphi\left(x,t\right)-\lambda f\left(\varphi\left(x,t\right)\right),\ x\in\mathbb{Q}_p^n,\ t\geq 0. \tag{10}$$

**We show, from a mathematical perspective, that the solutions of the Cauchy problem attached to equation (9) converge to the solutions of the Cauchy problem attached to equation (10), see Theorem 11, in the case that $f \in C^2$ with three zeros at $-1$, $0$, $1$.** Equation (10) is formally the $L^2$-gradient of the following energy functional:

$$E(\varphi) = \frac{1}{4}\int_{\mathbb{Q}_p^n}\int_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\{\varphi(x)-\varphi(y)\}^2 d^n x d^n y + \lambda \int_{\mathbb{Q}_p^n} W(\varphi(x))\, d^n x$$

where $\varphi$ is a scalar density function defined on $\mathbb{Q}_p^n$ that takes values in $[-1,1]$, $W$ is a double-well potential having minima at $\pm 1$ as before.

# Finite Approximations

We now study finite approximations to the solutions of

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + Au(x,t) = -\lambda f(u(x,t)), & x \in \mathbb{Q}_p^n, \quad t \geq 0 \\ u(x,0) = u_0(x), \end{cases} \tag{11}$$

where function $f(u)$ satisfies all the conditions given before.

# Finite Approximations

Our goal is to approximate the solution $u(x,t)$ of Cauchy Problem (11) in $X_\infty$ using only that $u_0(x) \in X_\infty$ and $-1 \leq u_0(x) \leq 1$. It is possible to approximate $u(x,t)$ without using any a priori information on the initial solution, however this requires to impose to the nonlinearity $f$ to be globally Lipschitz, this last condition reduces a lot the potentials $W$ to which we can apply our results.
The discretization of Cauchy problem (11) in the spaces $X_N$ takes the following form:

$$\begin{cases} \frac{d}{dt} u_N(t) + A_N u_N(t) = -\lambda P_N f(E_N u_N(t)) \\ u_N(0) = P_N u_0. \end{cases} \tag{12}$$

By taking $P_N u_0(x) = \sum_{\mathbf{i} \in G_N^n} u_0(\mathbf{i})\, \Omega\left(p^N \|x - \mathbf{i}\|_p\right)$ and identifying $u_N(t)$ with the column vector $[u_N(\mathbf{i}, t)]_{\mathbf{i} \in G_N^n}$, we can rewrite Cauchy problem (12) as

$$\begin{cases} \frac{d}{dt}[u_N(\mathbf{i}, t)]_{\mathbf{i} \in G_N^n} + A^{(N)}[u_N(\mathbf{i}, t)]_{I \in G_N^n} = -\lambda [f(u_N(\mathbf{i}, t))]_{\mathbf{i} \in G_N^n} \\ [u_N(\mathbf{i}, 0)]_{\mathbf{i} \in G_N^n} = [u_0(\mathbf{i})]_{\mathbf{i} \in G_N^n}, \end{cases} \tag{13}$$

cf. Lemma 6.

## Theorem

*(i)* $-A$ *is the generator of a strongly continuous semigroup* $\left\{e^{-tA}\right\}_{t\geq 0}$ *on* $X_\infty$. *Moreover,* $\left\|e^{-tA}\right\| \leq 1$ *for* $t \geq 0$ *and*

$$\lim_{N\to\infty} \sup_{t\geq 0} e^{bt} \left\| E_N e^{-A_N t} P_N \varphi - e^{-tA}\varphi \right\|_\infty = 0 \text{ for all } \varphi \in X_\infty,\ b \in (0,\infty).$$

*(ii) Take* $u_0(x) \in X_\infty$ *with* $-1 \leq u_0(x) \leq 1$. *Let* $u$ *be the solution of (11) and let* $u_N$ *be the solution of (12). Then*

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \left\| E_N u_N(t) - u(t) \right\|_\infty = 0.$$

Thanks for your kind attention !