RANDOM WALKS AND PERCOLATION IN A HIERARCHICAL LATTICE

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RANDOM WALKS

1. Degrees of transience and recurrence

Some aspects of the long-time behaviour of random walks (Euclidean and hierarchical) depend only on the degree.

$X \equiv \{X(t), t \geq 0\}$: Lévy process in a Polish space S with (additive) Abelian group structure. For countable S, X is a continuous-time random walk.
Semigroup of X:

$$T_t\varphi(x) = E_x\varphi(X(t)), \quad t \geq 0, \quad \varphi \in B_b(S),$$

$(B_b(S)$: bounded measurable functions with bounded support).

Potential (or Green) operator of X:

$$G\varphi = \int_0^\infty T_t\varphi dt, \qquad \varphi \in B_b(S),$$

X is *recurrent* if $G\varphi \equiv \infty$, $\varphi \in C_b^+(S)$, $\varphi \neq 0$.

Fractional powers of G:

$$G^{\zeta}\varphi = \frac{1}{\Gamma(\zeta)}\int_0^{\infty} t^{\zeta-1} T_t \varphi dt, \quad \zeta > 0,$$

Degree of X:

$$\gamma = \sup\{\zeta > -1 : G^{\zeta+1}\varphi < \infty \;\; \forall \varphi \in B_b^+(S)\},$$

γ is *degree of transience* if $\gamma > 0$.
$-\gamma$ is *degree of recurrence* if $-1 < \gamma < 0$.
If $\gamma < \infty$, it may be that $G^{\gamma+1}\varphi < \infty$: write γ^+, or $G^{\gamma+1}\varphi \equiv \infty \;\; (\varphi \neq 0)$: write γ^-.
A process with degree 0^- *is* called *critically recurrent.*
For integer $k \geq 1$, X is
k-strongly transient if $||G^{k+1}\varphi|| < \infty$, $\gamma > k$,
k-weakly transient if $||G^k\varphi|| < \infty$, $G^{k+1}\varphi \equiv \infty$, $\gamma \in (k-1, k)$,
$k = 1$: usual strong/weak transience.

The degree of transience is also given by

$$\gamma = \sup\{\zeta > 0, \quad EL_{B_R}^{\zeta} < \infty \quad \forall R > 0\},$$

L_{B_R} = last exit time from B_R,

(B_R: centered open ball of radius R).

The classification of transience is used for some stochastic models of multilevel branching particle systems, including hierarchically structured populations. The behaviour of occupation time fluctuations depends on relationships between levels of strong/weak transience and levels of branching.

2. Processes in Euclidean space

X : symmetric α-stable process in $\mathbb{R}^d, 0 < \alpha \leq 2$,
$\alpha = 2$: Brownian motion, or simple (continuous-time) random walk in $\mathbb{Z}^d$,

$$\gamma = \frac{d}{\alpha} - 1, \quad \gamma^-,$$

γ is restricted to values in $[-1/2, \infty)$,

$p_t(0,0) \sim$ const. $t^{-(\gamma+1)} =$ const. $t^{-d/\alpha}$ as $t \to \infty$,

recurrent: $d \leq \alpha$, transient: $d > \alpha$,
critically recurrent: $d = \alpha$ (1 or 2),
k-strongly transient: $\alpha < d/(k+1)$,
k-weakly transient: $d/(k+1) \leq \alpha < d/k$.

3. Hierarchical random walks

Hierarchical lattice (group) of order N (integer ≥ 2):

$$\Omega_N = \{x = (x_1, x_2, \dots) : x_i \in \mathbb{Z}_N, \sum_i x_i < \infty\} = \bigoplus_{i=1}^{\infty} \mathbb{Z}_N^i,$$

$\mathbb{Z}_N = \{0, 1, \dots, N-1\}$: cyclic group,
addition componentwise mod$N$. (Different from p-adics.)

Hierarchical distance:

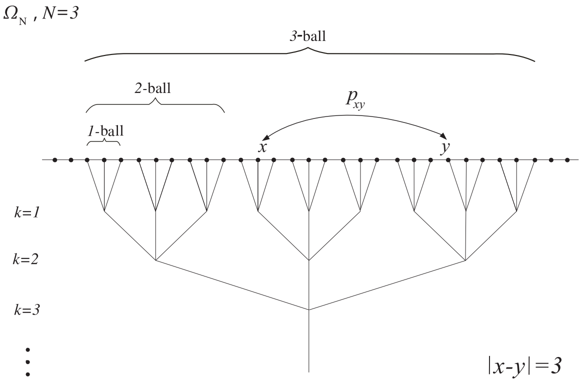
$$|x - y| = \begin{cases} 0 & \text{if } \quad x = y, \\ \max\{i, x_i \neq y_i\} & \text{if } \quad x \neq y, \end{cases}$$

satisfies the strong (non-Archimedean) triangle inequality:

$$|x - y| \leq \max\{|x - z|, |z - y|\} \quad \text{for any} \quad x, y, z.$$

$(\Omega_N, |\cdot|)$ is an ultrametric space. One point of Ω_N is designated as 0.

Ω_N has a tree representation:



Random walk: p_{xy} is the probability of jumping from x to y.
Percolation: p_{xy} is the probability of connection between x and y.
p_{xy} depends on the hierarchical distance $|x - y|$.

A *hierarchical random walk* $\xi = (\xi_n)_{n\geq 0}$ in Ω_N starting at 0 is defined by

$$\xi_0 = 0,\ \xi_n = \rho_1 + \cdots + \rho_n,\ n \geq 1,$$

where $\rho_1, \rho_2, \ldots$ are independent copies of ρ, which is a random element of Ω_N with distribution

$$P(\rho = y) = \frac{r_{|y|}}{N^{|y|-1}(N-1)}, \quad y \neq 0, \quad P(\rho = 0) = 0,$$

$(r_j)_{j\geq 1}$ is a probability law on $\{1, 2, \ldots\}$.

First, distance j is chosen with probability r_j, and then a point is chosen uniformly among the $N^{j-1}(N-1)$ points at distance j from the previous position of the walk. This gives Parisi matrix.
We call this, r_j-$r.w$. We assume there exists arbitrarily large j such that $r_j > 0$ (otherwise, by ultrametricity, the walk remains trapped in a ball).

n-step transition probability (by Fourier transform):

$$p^{(n)}(x,y) = (\delta_{0,|y|} - 1)\frac{f_{|y|}^n}{N^{|y|}} + (N-1)\sum_{k=|y|+1}^{\infty} \frac{f_k^n}{N^k},\ n \geq 1, y \in \Omega_N,$$

$$f_k = \sum_{j=1}^{k-1} r_j - \frac{r_k}{N-1} = 1 - r_k\frac{N}{N-1} - \sum_{j=k+1}^{\infty} r_j,\ k \geq 1,$$

$$p^{(1)}(0,0) = 0,$$

continuos-time random walk with unit rate holding time:

$$p_t(0,y) = e^{-t}\sum_{n=0}^{\infty} \frac{t^n}{n!} p^{(n)}(0,y), \quad t \geq 0,$$

$$p_t(0,y) = (\delta_{0,|y|} - 1)\frac{e^{-h_{|y|}t}}{N^{|y|}} + (N-1)\sum_{j=|y|+1}^{\infty} \frac{e^{-h_j t}}{N^j},\quad t \geq 0,$$

$$h_j = 1 - f_j = r_j\frac{N}{N-1} + \sum_{i=j+1}^{\infty} r_i, \quad j = 1, 2, \ldots.$$

General random walk: $(\mu, (c_j), N)$-*r.w.*

$$r_j = D\frac{c_{j-1}}{N^{(j-1)/\mu}}, \quad j = 1, 2, \ldots$$

$\mu > 0$, $c_j \geq 0$, $j = 0, 1, \ldots$, D: normalizing constant.

Example 1: $(1, (c^j), N)$-*r.w*, called c-*r.w.*
For a constant $0 < c < N$, the c-*r.w.* is defined by

$$r_j = \left(1 - \frac{c}{N}\right)\left(\frac{c}{N}\right)^{j-1}, \quad j = 1, 2, \ldots,$$

$$\gamma = \frac{\log c}{\log(N/c)}, \quad \gamma^{-}$$, γ takes any value in $(-1, \infty)$,

recurrent: $c \leq 1$, (and null recurrent, return time has infinite expectation), critically recurrent: $c = 1$, transient: $c > 1$,

k-strongly transient: $c > N^{(k+1)/(k+2)}$,
k-weakly transient: $N^{k/(k+1)} < c \leq N^{(k+1)/(k+2)}$.

$$p_t(0,0) \sim \text{const. } t^{-(\gamma+1)} h_t = \text{const. } t^{-\log N/\log(N/c)} h_t \quad \text{as } t \to \infty,$$

$$h_t = \sum_{j=-\infty}^{\infty} (ba^j t)^{\gamma+1} e^{-ba^j t}, \quad t > 0,$$

$$\gamma = \frac{\log c}{\log(N/c)},$$

$$a = N^{-1/(\gamma+1)}, \quad b = \frac{N^{(\gamma+2)/(\gamma+1)} - 1}{N-1},$$

The function h_t is periodic in logarithmic scale ($h_t = h_{at}, t > 0$), oscillates between two positive values, slower as $t \to \infty$ and faster as $t \to 0$.

(Comparing with Ogielski-Stein, 1985, $N = 2$, $T/\Delta" = "1/\log(N/c)$.)

The c-*r.w.* is a caricature of the α-stable process on $\mathbb{R}^d$, $d > \alpha$. The potential kernel of the c-*r.w.* is

$$G_{N,\gamma}(x) = \text{const. } N^{-|x|\gamma/(\gamma+1)}.$$

Equating the γ's for α-stable and c-*r.w.*:

$$\frac{d}{\alpha} - 1 = \frac{\log c}{\log(N/c)}, \quad c = N^{1-\alpha/d},$$

$$G_{N,\gamma}(x) = \text{const. } \rho(x)^{-(d-\alpha)},$$

$\rho(x) = N^{|x|/d}$ (Euclidean radial distance from 0 to x, the volume of a ball of radius ρ grows like ρ^d), $\rho(x)$ is also an ultrametric, $G_{N,\gamma}(x)$ has the same decay for α-stable process in $\mathbb{R}^d$ and *c-r.w.*

c can be chosen so that the c-*r.w.* simulates an α-stable process in $\mathbb{R}^d$ for any dimension d, even non-integer (e.g. $d = 4 \pm \varepsilon$).

For the $(\mu, (c^j), N)$-*r.w.* with $0 < c < N^{1/\mu}$,

$\gamma_N = \mu - 1 + O(1/\log N)$,

mean field limit: $\gamma_N \to \mu - 1$ as $N \to \infty$.

For $\mu = 1$, Brownian motion in dimension 2.

For $\mu = 2$, Brownian motion in dimension 4.

Example 2: (μ, j^β, N)-*r.w.*, $\mu > 0$, $\beta \geq 0$.

$$r_j = D\frac{(j+1)^\beta}{N^{(j-1)/\mu}}, \quad j = 1, 2, \ldots$$

$$\gamma = \mu - 1, \quad \gamma^+ \text{ if } \beta > \frac{1}{\mu}, \quad \gamma^- \text{ if } \beta \leq \frac{1}{\mu},$$

For this and other $(\mu, (c_j), N)$-*r.w.* such that $\lim_{j\to\infty} \frac{c_{j+1}}{c_j} = 1$,

for each $\varepsilon > 0$,

$$p_t(0,0) = o(t^{-\mu+\varepsilon}) \text{ as } t \to \infty,$$

$$(p_t(0,0))^{-1} = o(t^{\mu+\varepsilon}) \text{ as } t \to \infty.$$

Remark on Q-matrix:

X : continuous-time Markov chain, countable state space,
transition matrix (jump probabilities): $P = (p(x, y))$,
n-step transition matrix: $P^n = (p^{(n)}(x, y))$,
continuous-time transition matrix:
$P_t = (p_t(x, y))$,

$$p_t(x, y) = P_x(X(t) = y) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} p^{(n)}(x, y), \qquad (a)$$

$\lambda = 1$ if all states have unit rate holding times,

in general each state x can have its own holding time (exponential distribution with mean $1/q_x$), the *Q-matrix* (*generator*) $Q = (q(x, y))$ is given by

$$q(x,y) = \lim_{t \searrow 0} \frac{1}{t}(p_t(x,y) - \delta_{xy}) = \begin{cases} q_x p(x,y), & y \neq x, \\ q(x,x), & y = x, \end{cases}$$

where

$q(x,x) = -\sum_{z \neq x} q(x,z)$ (assumed finite),

then (a) holds with $\lambda = \sup_x |q(x,x)| \quad (< \infty)$,

$$P = \frac{1}{\lambda} Q + I, \quad P_t = e^{Qt},$$

if $q_x = q \quad \forall x$, then for each x, $p(x,x) = 0$ iff $q(x,x) = -\lambda$, $p(x,x) > 0$ iff $|q(x,x)| < \lambda$.

4. Distance Markov chain

$(\xi_n)_{n\geq 0}$: r_j-*r.w.* in Ω_N. The *distance Markov chain* $(Z_n)_{n\geq 0}$ is defined by $Z_n = |\xi_n|$.

The transition probability $p_{ij} = P[Z_{n+1} = j | Z_n = i]$ is given by

$$p_{ij} = \begin{cases} r_j, & j > i, \\ r_1 + \cdots + r_{i-1} + r_i \dfrac{N-2}{N-1}, & j = i \ \ (\neq 0), p_{00} = 0, \\ r_i \dfrac{1}{N^{i-j}}, & 0 < j < i, \\ r_i \dfrac{1}{N^{i-1}(N-1)}, & 0 = j < 1, \end{cases}$$

The c-*r.w.* tends to stay at the same distance from 0 for a long time, and the value of c determines the tendency to go away or towards 0.

Let $D_i = \sum_{j=1}^{\infty} j p_{ij}$ (expected distance from 0 after one step from i),
For the c-*r.w.* with $c = 1$,

$$D_i = i + \frac{1}{N^{2i-1}(N-1)}, \quad i > 0,$$

(Z_n) has positive drift, but it is recurrent.

(Z_n) is analogous to a Bessel process, but there is a qualitative difference.
If $c \geq 1$, (Z_n) has positive drift, it is a submartingale. If $c < 1$, (Z_n) behaves like a submartingale up to distance

$$L_N(c) = \frac{1}{\log N}\left(-\log\left(1 - c\left(\frac{N-1}{N-c}\right)^2\right)\right),$$

and when it exceeds this distance the drift becomes negative.
$(L_N(c) \to \infty$ as $c \nearrow 1)$.

Maximal distance in n steps:
for $\delta > 1/\log(N/c)$,

$$P\left(\max_{1\leq k\leq n} Z_k \geq \delta \log n\right) \sim \frac{\lfloor \delta \log(c/N) \rfloor}{1 + \lfloor \delta \log(c/N) \rfloor} n^{1+\lfloor \delta \log(c/N) \rfloor} \text{ as } n \to \infty,$$

for $\delta > 2/\log(N/c)$,

$$P\left(\max_{1\leq k\leq n} Z_k \geq \delta \log n \ \text{ i.o.}\right) = 0.$$

Consistent with Ogielski-Stein.

Separation of time scales:

For the $(\mu, (c^j), N)$-*r.w.* with $\mu \geq 1$, $k \geq 1$,

$$\lim_{N\to\infty} P\left(\max_{1\leq j\leq \lfloor N^{k/\mu}\rfloor} Z_j = \ell\right) = \begin{cases} e^{-c^k}, & \ell = k, \\ 1 - e^{-c^k}, & \ell = k+1, \end{cases}$$

$N^{k/\mu}$ is the right time scale for observing the exit behaviour of the walk from a closed ball of radius k. As $N \to \infty$, only the ball of radius k and the surrounding ball of radius $k+1$ are relevant.

This is used for separation of time scales in branching systems.

5. Occupation time fluctuations

Infinite system of particles in Ω_N. The particles perform independent hierarchical random walks in continuous time. In addition the particles can branch at one or more levels.

If X_t denotes the empirical measure on Ω_N describing the configuration of the system at time t, one problem is to find an appropriate norming a_t such that the occupation time fluctuation

$$\frac{1}{a_t}\int_0^t (X_s - EX_s)ds$$

has a non-trivial limit (in distribution) as $t \to \infty$.

A key role is played by the growth as $t \to \infty$ of the incomplete potential operator

$$G_t = \int_0^t T_s ds$$

and its powers (G_t^k for k-level branching).

If the growth is given by an increasing function f_t, then $a_t = \left(\int_0^t f_s ds \right)^{1/2}$.

For the c-$r.w.$,

$$a_t = \begin{cases} (t^{k+1-\gamma} h_t^{(k)})^{1/2} & \text{for} \quad k-1 < \gamma < k, \\ (t \log t)^{1/2} & \text{for} \quad \gamma = k, \\ t^{1/2} & \text{for} \quad \gamma > k, \end{cases}$$

where $h_t^{(k)}$ is a periodic function in logarithmic scale.

For the $(k+1, j^\beta, N)$-$r.w$, $k = 0, 1, 2$,

$$a_t = \begin{cases} t^{1/2} (\log t)^{(1-(k+1)\beta)/2} & \text{for} \quad \beta < 1/(k+1), \\ (t \log \log t)^{1/2} & \text{for} \quad \beta = 1/(k+1), \\ t^{1/2} & \text{for} \quad \beta > 1(k+1). \end{cases}$$

A suitably time-scaled and normalized occupation time fluctuation limit of the system without branching in the case $c < 1$ ($\gamma < 0$) leads to a long-range dependent centered Gaussian process $\xi = (\xi_t)_{t \geq 0}$ with covariance function

$$Cov(\xi_s, \xi_t) = \frac{1}{2}(s^{2H} h_s + t^{2H} h_t - |s-t|^{2H} h_{|s-t|}), \quad s, t \geq 0,$$

$$h_t = \sum_{j=-\infty}^{\infty} (ba^j t)^{-2H} (e^{-ba^j t} - 1 + ba^j t), \quad t > 0, \quad h_0 = 0,$$

$$H = \frac{1}{2}(1-\gamma) \in \left(\frac{1}{2}, 1\right), \text{ constants } b > 0 \text{ and } a \in (0,1).$$

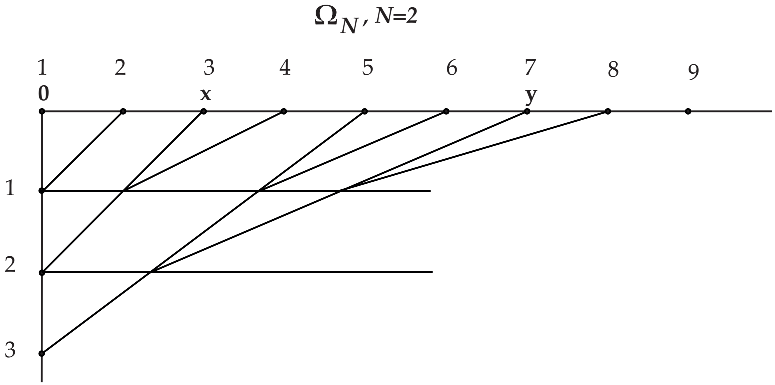
The function h_t is periodic in logarithmic scale ($h_t = h_{at}, t > 0$), oscillates between two positive values, slower as $t \to \infty$ and faster as $t \to 0$. This is an oscillatory analogue of fractional Brownian motion with Hurst parameter H ($h_t \equiv 1$).

A related system with branching gives an oscillatory analogue of sub-fractional Brownian motion.

Fractional and sub-fracional Brownian motion occur in the Euclidean case (α-stable processes).

6. Mixing time

Enumerate Ω_N by $\mathbb{N} = \{1, 2, \ldots\}$.



The transition matrix $P = (p(i,j))_{i.j=1,2,\ldots}$ has Parisi from

$$M_1 = \begin{bmatrix} 0 & & \frac{r_1}{N-1} \\ & \ddots & \\ \frac{r_1}{N-1} & & 0 \end{bmatrix}_{N \times N},$$

$$M_k = \begin{bmatrix} M_{k-1} & & \frac{r_k}{N^{k-1}(N-1)} \\ & \ddots & \\ \frac{r_k}{N^{k-1}(N-1)} & & M_{k-1} \end{bmatrix}_{N^k \times N^k}, \; k = 2, 3, \ldots,$$

$P = M_\infty$.

Restrict the walk to hierarchical distance n, $P = M_n$, and assume it is normalized, $\sum_{k=1}^{n} r_k^{(n)} = 1$.
m-step transition matrix: $P^m = (p^{(m)}(i,j))_{i,j=1,\dots,N^n}$.
The walk has stationary distribution $\Pi(j) = 1/N^n$, $j = 1,\dots,N^n$, and $p^{(m)}(i,j) \to \Pi(j)$ as $m \to \infty$ $\forall i,j$.

The mixing time measures how fast the walk approximates the stationary distribution.
For $\varepsilon > 0$ (small), the *mixing time* is defined by

$$t_{\mathrm{mix}}(\varepsilon) = \min_m \left\{ \max_i ||p^{(m)}(i,\cdot) - \Pi||_{TV} \le \varepsilon \right\},$$

where

$$||p^{(m)}(i,\cdot) - \Pi||_{TV} = \max\{|p^{(m)}(i,A) - \Pi(A)|, A \subseteq \{1,\dots,N^n\}\}$$

(total variation distance).

From a general result on Markov chains,

$$t_{\mathrm{mix}}(\varepsilon) \leq \log\left(\frac{N^n}{\varepsilon}\right)\frac{1}{1-\lambda_*}$$

where λ_* is the next largest eigenvalue of P (the largest is 1). The number $1-\lambda_*$ is the *spectral gap*, $1/(1-\lambda_*)$ is the *relaxation time*. Similarly with continuous time.

For the $r_j^{(n)}$-*r.w.*,

$$\lambda_* = \sum_{k=1}^{n-1} r_k^{(n)} - \frac{r_n^{(n)}}{N-1} = 1 - \frac{N}{N-1} r_n^{(n)}, \text{ (multiplicity } N-1),$$

(the expression for λ_* may be negative because the walk is periodic. With the "lazy version" of the walk, which has transition matrix $\frac{1}{2}(P+I)$, all eigenvalues are non-negative).

At step $t_{\mathrm{mix}}(\varepsilon)$, the walk (restricted to distance n) has visited all the states about the same number of times.

7. More general hierarchical random walks

Replace Ω_N by the random hierarchical group

$\bigoplus_{i=1}^{\infty} \mathbb{Z}_{N_i}^{i}$, (N_i): random variables (i.i.d.),
and a random walk with random (r_j),
criterion for recurrence (from Flatto-Pitt, 1974):

$$\sum_{n=1}^{\infty} 1/M_n f_n = \infty,$$

where $f_n = \sum_{j=n}^{\infty} r_j$, $M_n = \prod_{i=1}^{n-1} N_i$, (assuming $\sup_i N_i < \infty$).

How to study the random series?, are there new phenomena?

Random walks on hierarchical graphs that are not groups are also studied.

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PERCOLATION

1. Percolation in Ω_N

The probability p_{xy} of connection between points x and y in Ω_N depends on the hierarchical distance $|x - y|$. All connections are independent.

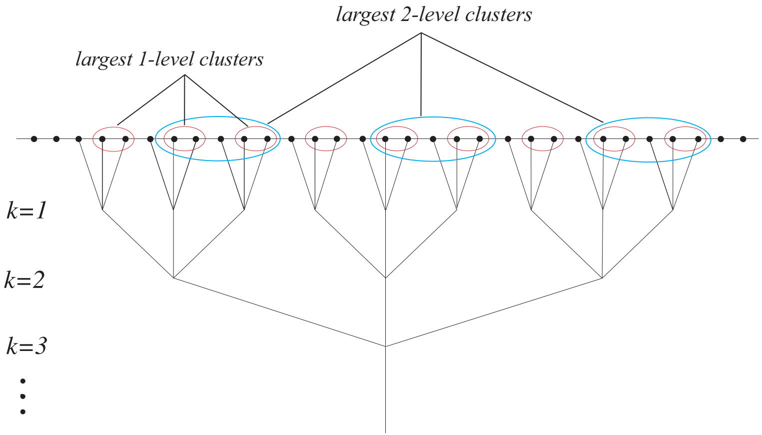
For each $k \geq 1$, in each k-ball (ball of diameter k) there is a largest *cluster*, i.e., a set of points any two of which are linked by a path of connections within the k-ball. If there are more than one, we choose one uniformly.

The question about percolation is if a given point of Ω_N (e.g. 0) belongs to an infinite cluster with positive probability. This involves all hierarchical distances.

Percolation in Ω_N is necessarily long range percolation (due to the ultrametric structure).

The repeating pattern of connections at each level k lends itself to a renormalization approach (more easily than for percolation on $\mathbb{Z}^d$).

Ω_N , $N=3$



2. Mean field percolation ($N \to \infty$)

The probability p_{xy} of connection between x and y such that $|x-y| = k \geq 1$ is given by

$$\frac{c_k}{N^{2k-1}}, \quad c_k > 0.$$

Definition: There is a *mean field percolation* if

$P_{\text{perc}} := \inf_k \liminf_{N\to\infty} P$ (0 is linked by a path of connections to a point at distance k) > 0. P_{perc} is the *probability of percolation*.

The probability that two different $(k-1)$-balls in a k-ball are connected is

$$p^{N,k} = 1 - \left(1 - \frac{c_k}{N^{2k-1}}\right)^{N^{2(k-1)}}, \quad k > 1, \quad p^{N,1} = \frac{c_1}{N}.$$

$$p^{N,k} = \frac{c_k}{N} + o\left(\frac{1}{N}\right) \quad \text{as} \quad N \to \infty, \quad k > 1.$$

The N $(k-1)$-balls in a k-ball constitute a random graph $G(N, p^{N,k})$, which is approximately an Erdős-Rényi random graph $G(N, c_k/N)$. Therefore it is possible to use the E-R theory of giant components.

E-R theory: Graph $G(N, c/N)$, N vertices, connection probability c/N.

If $c < 1$, the size of the largest connected component is of order $\log N$ as $N \to \infty$.

If $c > 1$, there is a unique largest component, called giant component, of size asymptotically βN as $N \to \infty$, where $\beta \in (0, 1)$ is the solution of

$$1 - \beta = e^{-c\beta},$$

and all other components are of order at most $\log N$.

Instead of the connection probabilites between $(k-1)$-balls in a k-ball, we consider the connection probabilities between their giant components. The probabilities of connections are random because the giant components have random sizes. Other connections become negligible as $N \to \infty$.

The vertices of the random graphs at each level k are the giant components at the previous level $k-1$, and each vertex has an internal structure determined by the giant components it contains in all the previous hierarchical levels, and the connections among them.

For each $k \geq 1$, let $\beta_k \in (0, 1)$ satisfy the nonlinear iteration

$$\beta_k = 1 - e^{-c_k \beta_{k-1}^2 \beta_k}, \quad \beta_0 = 1,$$

where $c_k \beta_{k-1}^2 > 1$,
(the β_{k-1}^2 in the equation for level k comes from the sizes of the giant components at the previous level $k - 1$.)

Theorem: Assume $c_k \nearrow \infty$ as $k \to \infty$, $c_1 > 2 \log 2$, and $c_2 > 8 \log 2$. Then there is mean field percolation if and only if $\sum_{k=1}^{\infty} e^{-c_k} < \infty$, and the probability of percolation in given by

$$P_{\text{perc}} = \prod_{k=1}^{\infty} \beta_k.$$

Percolation takes place along a cascade of giant components.

Example: Let $c_k = a \log k$, $a > 0$, for large k. There is mean field percolation if and only if $a > 1$.

3. Percolation with fixed N

It is not possible to use giant components.
The probability p_{xy} of connection between x and y such that $|x - y| = k$ is given by

$$\frac{c_k}{N^{(1+\delta)k}}, \quad c_k > 0, \quad \delta > 0.$$

Definition: Percolation occurs if there is a positive probability that a given point (e.g. 0) belongs to an infinite cluster (*percolation cluster*).

With fixed N we can only obtain sufficient conditions for percolation or for non-percolation, and there are open questions.

Non-critical cases: $\delta > 1$ and $\delta < 1$. Critical case: $\delta = 1$.

Theorem:

(a) If $\delta > 1$ and $\sup_k c_k < \infty$, then there is ultimate trapping in a ball, so percolation does not occur.
(b) If $\delta < 1$ and $c = \inf_k c_k$ is large enough, then percolation occurs through a chain (cascade) of clusters in k-balls, and the percolation cluster is unique.

Critical case: $\delta = 1$,
$c_k = C_0 + C_1 \log k + C_2 k^\alpha$,
$C_0, C_1, C_2 \geq 0$, $\alpha > 0$.
Consider the case $C_2 > 0$ ($C_0 = C_1 = 0$).

Theorem: For C_2 sufficiently large, there exists a unique percolation cluster of positive density $\geq \varepsilon(N, C_2, \alpha)$ to which 0 belongs with positive probability.

Main ideas of the proof:

An *ultrametric random graph* $\mathrm{URG}(N, |\cdot|)$ is a graph with N vertices and an ultrametric $|\cdot|$, and for each pair (x, y) there is a connection with probability p_{xy} which is a random variable that depends on $|x - y|$.
We consider the infinite graph $G(V_\infty, \mathcal{E}_\infty)$ with vertices $V_\infty = \Omega_N$ and edges $\mathcal{E}_\infty$ such that

$$p_{xy} = P((x, y) \in \mathcal{E}_\infty) = \frac{C_2 k^\alpha}{N^{2k}} \text{ if } |x - y| = k.$$

We want to know if $G(V_\infty, \mathcal{E}_\infty)$ has an infinite component (percolation cluster).

Let $\mathcal{C}_k$ denote the maximal connected component (cluster) in a k-ball, its density is

$$X_k = \frac{|\mathcal{C}_k|}{N^k}.$$

We consider for each $k \geq 1$ the ultrametric random graph

$$G_k(N, \{X_{k-1}(i),\ i = 1, \ldots,\ N\}),$$

where $X_{k-1}(i), i = 1, \ldots, N$, are the densities of the clusters of the N $(k-1)$-balls in the k-ball with center 0, and the probability of connection between two vertices is the probability that the clusters of the corresponding $(k-1)$-balls are connected. These probabilities are random, and they are independent conditionally on the densities.

If the density of the k-ball with center 0 has a positive liminf as $k \to \infty$, then $G(V_\infty, \mathcal{E}_\infty)$ has a percolation cluster.

$$X_k = \frac{1}{N^k} \sum_{i \in V_k} N^{k-1} X_{k-1}(i) = \frac{1}{N} \sum_{i \in V_k} X_{k-1}(i),$$

where V_k is the set of $(k-1)$-balls that are all connected.

$\mu_k =$ distribution of X_k,

$\mu_k = \Phi_k(\mu_{k-1})$, where $\Phi_k : P([0,1]) \to P([0,1])$ is the *renormalization transformation*.

If suffices to show that there exists $a > 0$ such that

$\liminf_{k \to \infty} P(X_k > a) > 0.$

Using Φ_k, it is shown that there exists $a > 0$ such that as $k \to \infty$, $E(X_k) \to a$ and $\mathrm{Var}(X_k) \to 0$, consequently, $\mu_k \Rightarrow \delta_a$.

4. Transience and recurrence of simple random walks on percolation clusters of Ω_N

Background on $\mathbb{Z}^d$:
Simple symmetric random walk (SRW) is recurrent for $d = 1, 2$, transient for $d \geq 3$ (Polya, 1921). For SRW on the (Bernoulli bond) percolation cluster of $\mathbb{Z}^d$ (assuming it exists), the transience/recurrence results are the same as for all $\mathbb{Z}^d$ (Grimmett-Kesten-Zhang, 1993). Since the percolation cluster is a disordered medium, traditional methods cannot be used. The appropriate method is electric circuit theory (Doyle-Snell, 1984).

Long-range percolation on $\mathbb{Z}^d$ and SRW on percolation clusters (Schulman, 1983, Newman-Schulman, 1986, Aizenman-Newman, 1986; SRW, Berger, 2002).
Probability of connection $p_{xy} = 1 - \exp(-\beta/|x-y|^s) \sim \beta/|x-y|^s$, large $|x-y|$.

Results for $d = 1$: If $1 < s \leq 2$, there can be percolation by increasing the probability of connection for (x, y) such that $|x - y| = 1$. If $s > 2$ there is no percolation. SRW on the cluster is transient if $1 < s < 2$, recurrent if $s = 2$.

Results for SRW on the percolation cluster of Ω_N with

$$p_{xy} = \frac{c_k}{N^{(1+\delta)k}} \text{ if } |x - y| = k.$$

Theorem: For almost all realizations of the cluster,

1. If $\delta < 1$, the SRW on the percolation cluster is transient.

2. In the critical case, $\delta = 1$, $c_k = C_2 k^{\alpha}$, there exists a critical value $\alpha_c \in (0, \infty)$ such that the SRW on the cluster is recurrent for $\alpha < \alpha_c$ and transient for $\alpha > \alpha_c$.

Remarks:

1) SRW on the percolation cluster is well-defined because each point of Ω_N has a finite number of neighbours.

2) Comparison with SRW on long-range percolation cluster of $\mathbb{Z}$: Using the ultrametric $N^{|x-y|}$ on Ω_N, and $s = \delta + 1$, the results agree for $1 < s < 2$, $\delta < 1$. For $\delta = 1$ and $s = 2$ there is no comparison because in the Euclidean model β does not depend on distance, and in the ultrametric model c_k depends on distance k.

3) Open questions: From the proof we know that $\alpha_c \in (1, 6)$. What is the value of α_c?, is the SRW transient or recurrent for $\alpha = \alpha_c$?

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