New applications of the p -adic Nevanlinna Theory

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1. Introduction

Let $I\!K$ be a complete ultrametric algebraically closed field of characteristic 0 whose ultrametric absolute value is denoted by $| \,.\, |$ (we denote by $| \,.\, |_\infty$ the Archimedean absolute value of $I\!R$). The Nevanlinna theory, well known for complex meromorphic functions [18], was examined over $I\!K$ by Ha Huy Khoai and was finally constructed by A. Boutabaa. Next, a similar theory was made for unbounded meromorphic functions in an "open" disk of $I\!K$, taking into account Lazard's problem. In 2007, M. O. Hanyak and A. A. Kondratyuk constructed a Nevanlinna theory for meromorphic functions in a *punctured complex plane*, i.e., in the set $\mathbb{C} \setminus \{a_1, ..., a_m\}$, where we understand that the meromorphic functions can admit essential singularities at $a_1, ..., a_m$.

Here we recall the Nevanlinna theory for meromorphic functions in the complement of an open disk by using specific properties of the Analytic Elements on infraconnected subsets of $\mathbb{K}$ and particularly the Motzkin Factorization. We can also obtain a Nevanlinna theory on three small functions, as it was done in the classical context. Once the Nevanlinna Theory is established for such functions, we can apply it to obtain results on uniqueness and branched values as it was done in similar problems. Next we describe new results on meromorphic functions sharing two sets and particular properties of meromorphic functions of the form $f^n(x)f^m(ax+b)$ with regards to branched values and Picard's values.

Notation: Given $r > 0$, $a \in \mathbb{K}$ we denote by $d(a, r)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$, by $d(a, r^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$, and by $C(a, r)$ the circle $\{x \in \mathbb{K} \mid |x - a| = r\}$. Given $r'' > r'$, we put $\Delta(0, r', r'') = d(0, r'') \setminus d(0, r'^-)$.

Henceforth, we fix $R > 0$, we denote by S the disk $d(0, R^-)$ and put $D = \mathbb{K} \setminus S$.

Given a bounded function f in D, we put $\|f\| = \sup_D |f(x)|$. Given a subset E of $\mathbb{K}$ having infinitely many points, we denote by $R(E)$ the $\mathbb{K}$-algebra of rational functions $h \in \mathbb{K}(x)$ having no pole in E.

We then denote by $H(E)$ the $\mathbb{K}$-vector space of analytic elements on E, i.e., the completion of $R(E)$ with respect to the topology of uniform convergence on E. By classical properties of analytic elements, we know that given a circle $C(a,r)$ and an element f of $H(C(a,r))$, i.e., a Laurent series $f(x) = \sum_{-\infty}^{+\infty} c_n(x-a)^n$ converging whenever $|x| = r$, then $|f(x)|$ is equal to $\sup_{n\in\mathbb{Z}} |c_n|r^n$ in all classes of the circle $C(a,r)$ except maybe in finitely many. When $a = 0$, we put $|f|(r) = \sup_{n\in\mathbb{Z}} |c_n|r^n$. Then $|f|(r)$ is a multiplicative norm on $H(C(0,r))$.

We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{A}(d(a, R^-)$ the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converging in all $d(a, R^-)$ and by $\mathcal{A}(D)$ the $\mathbb{K}$-algebra of Laurent series $\sum_{-\infty}^{\infty} c_n(x-a)^n$ converging in D. Similarly, we will denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$, by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions in $d(a, R^-)$ i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$, and by $\mathcal{M}(D)$ the field of meromorphic functions in D i.e. the field of fractions of $\mathcal{A}(D)$.

Next, we will denote by $\mathcal{A}_b(d(a, R^-))$, the set of $f \in \mathcal{A}(d(a, R^-))$ that are bounded in $d(a, R^-)$ and we put $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. We will denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$ and put $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

We will denote by $\mathcal{A}^w(D)$ the set of $f \in \mathcal{A}(D)$ admitting finitely many zeros in D and we put $\mathcal{A}^*(D) = \mathcal{A}(D) \setminus \mathcal{A}^w(D)$ and similarly, we denote by $\mathcal{M}^w(D)$ the field of fraction of $\mathcal{A}^w(D)$ and we put $\mathcal{M}^w(D) = \mathcal{M}(D) \setminus \mathcal{M}^*(D)$. So, $\mathcal{M}^*(D)$ is the set of meromorphic functions in D having at least infinitely many zeros or infinitely many poles in D.

2. Meromorphic functions

In this paragraph, we will recall basic properties of meromorphic functions.

Let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(D)$). Given $r < R$ (resp. $r > R$), we know that $|f(x)|$ admits a limit denoted by $|f|(r)$ when $|x|$ tends to r while remaining different from r.

Let $f \in \mathcal{A}(d(0, R^-))$ (resp. $f \in \mathcal{A}(D)$) and let $\alpha \in d(a, R^-)$, resp. $\alpha \in D$). If f admits a zero of order q at α we set $\omega_\alpha(f) = q$ and if $f(\alpha) \neq 0$, we set $\omega_\alpha(f) = 0$.

Let $f = \dfrac{h}{l} \in \mathcal{M}(d(a, R^-))$, (resp. $f \in \mathcal{M}(D)$). For each $\alpha \in \mathbb{K}$ (resp. $\alpha \in d(a, R^-)$, resp. $\alpha \in D$) the number $\omega_\alpha(h) - \omega_\alpha(l)$ does not depend on the functions h, l choosed to make $f = \dfrac{h}{l}$. Thus, we can generalize the notation by setting $\omega_\alpha(f) = \omega_\alpha(h) - \omega_\alpha(l)$.

If $\omega_\alpha(f)$ is an integer $q > 0$, α is called *a zero of f of order q.*
If $\omega_\alpha(f)$ is an integer $q < 0$, α is called *a pole of f of order $-q$.*
If $\omega_\alpha(f) \geq 0$, f will be said to be *holomorphic* at α.

Definition and notation: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$ have a pole α of order q and let $f(x) = \sum_{k=-q}^{-1} a_k(x-\alpha)^k + h(x)$ with $a_{-q} \neq 0$ and $h \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$ and h holomorphic at α.

Accordingly to usual notations the coefficient a_{-1} is called *residue of f at α* and denoted by $\text{res}(f, \alpha)$.

It seems obvious that the condition for a meromorphic function to admit primitives is that all residues are null. Actually, the proof is not this immediate.

Theorem 2.2: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-)$, resp. $f \in \mathcal{M}(D)$). Then f admits primives if and only if all residues of f are null.*

Definitions: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}(D)$) and let $b \in \mathbb{K}$. Then b will be said to be *an exceptional value for f* if $f - b$ has no zero in $\mathbb{K}$ (resp. in $d(a, R^-)$, resp. in D)). Moreover, if $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. if $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}(D)$)), b will be said to be *a quasi-exceptional value for f* if $f - b$ has finitely many zeros.

Theorem 2.3: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$, (resp. $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}^*(D)$)). Then f admits at most one quasi-exceptional value. Moreover, if f has finitely many poles in $\mathbb{K}$ (resp. in $d(a, R^-)$, resp. in D), then f has no quasi-exceptional value.*

The following theorem 2.4 was proven by Jean-Paul Bézivin in a joint work with Kamal Boussaf and the first author:

Theorem 2.4 (J.P. Bézivin): *Let $f \in \mathcal{M}(\mathbb{K})$ and for each $r > 0$, let $\gamma(f, r)$ be the number of multiple poles of f in $d(0, r)$. If there exists $c > 0$ and $s \in \mathbb{N}$ such that $\gamma(r, f) \leq cr^s \ \forall r > 1$, then f' admits no quasi-exceptional value.*

Theorem 2.4 suggests the following conjecture that is already proven in a wide domain:

Conjecture : *Given $f \in \mathcal{M}(\mathbb{K})$, then f' admits no quasi-exceptional value.*

3. Nevanlinna Theory in the classical p-adic context

The Nevanlinna Theory was made by Rolf Nevanlinna on complex functions in the 1920th. It consists of defining counting functions of zeros and poles of a meromorphic function f and giving an upper bound for multiple zeros and poles of various functions $f - b,\ b \in \mathbb{C}$.

A similar theory for functions in a p-adic field was constructed by A. Boutabaa, after some previous works by Ha Huy Khoai. The p-adic Nevanlinna Theory was first stated and correctly proved by Abedelbaki Boutabaa in $\mathcal{M}(\mathbb{K})$ in 1988. The theory was extended to functions in $\mathcal{M}(d(0, R^-))$ by taking into account Lazard's problem in 1999.

Throughout the next paragraphs, we denote by I the interval $[t, +\infty[$, by J an interval of the form $[t, R[$ with $t > 0$ and by L the interval $[R, +\infty[$.

We denote by f a function that belongs either to $\mathcal{M}(\mathbb{K})$ or to $\mathcal{M}(S)$.

We have to introduce the counting function of zeros and poles of f, counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let $(a_n),\ 1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n. We set

$$Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$$

and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity.*

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r,f)$ the counting function of zeros of f without multiplicity:

$$\overline{Z}(r,f) = \overline{\omega_0}(f)\log r + \sum_{n=1}^{\sigma(r)}(\log r - \log|a_n|) \text{ and so,}$$

$\overline{Z}(r,f)$ is called *the counting function of zeros of f in $d(0,r)$ ignoring multiplicity.*

In the same way, considering the finite sequence (b_n), $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n, we put

$$N(r,f) = \max(-\omega_0(f),0)\log r + \sum_{n=1}^{\tau(r)} t_n(\log r - \log|b_n|)$$

and then $N(r,f)$ is called *the counting function of the poles of f, counting multiplicity.*

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega_0}}(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega_0}}(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$$\overline{N}(r,f) = \overline{\overline{\omega_0}}(f)\log r + \sum_{n=1}^{\tau(r)}(\log r - \log|b_n|) \text{ and then}$$

$\overline{N}(r,f)$ is called *the counting function of the poles of f, ignoring multiplicity.*

Now, we can define the Nevanlinna function $T(r, f)$ in I or J as $T(r, f) = \max(Z(r, f), N(r, f))$ and the function $T(r, f)$ is called *characteristic function of f or Nevanlinna function of f*.

Finally, if Y is a subset of $\mathbb{K}$ we will denote by $Z^Y(r, f')$ the counting function of zeros of f', excluding those which are zeros of $f - a$ for any $a \in Y$.

Remark: If we change the origin, the functions Z, N, T are not changed, up to an additive constant.

Theorem 3.1: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) have no zero and no pole at 0. Then*

$$\log(|f|(r)) = \log(|f(0)|) + Z(r, f) - N(r, f).$$

Theorem 3.2 (First Main Theorem) : *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}(S)$). Then*

$Z(r, fg) \leq Z(r, f)+Z(r, g),\ N(r, fg) \leq N(r, f)+N(r, g),$

Then $T(r, f+b) = T(r, f)+O(1)$, $T(r, fg) \leq T(r, f)+T(r, g)$, $T(r, f+g) \leq T(r, f)+T(r, g)+O(1)$, $T(r, cf) = T(r, f)\ \forall c \in \mathbb{K}^*,\ T(r, \dfrac{1}{f}) = T(r, f))$,

$$T(r, \frac{f}{g}) \leq T(r, f)) + T(r, g).$$

Let $P(X) \in \mathbb{K}[X]$. *Then* $T(r, P(f)) = \deg(P)T(r, f)+O(1)$ *and* $T(r, f'P(f) \geq T(r, P(f))$.

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{A}(S)$*). Then* $Z(r, fg) = Z(r, f)+Z(r, g),\ T(r, f) = Z(r, f))$ $T(r, fg) = T(r, f) + T(r, g) + O(1)$ *and* $T(r, f + g) \leq \max(T(r, f), T(r, g))$.

Moreover, if $\lim\limits_{r\to+\infty} T(r, f) - T(r, g) = +\infty$ *then* $T(r, f + g) = T(r, f)$ *when* r *is big enough.*

Theorem 3.3: *Let* $f \in \mathcal{M}(\mathbb{K})$. *Then* f *belongs to* $\mathbb{K}(x)$ *if and only if* $T(r, f) = O(\log r)$.

Corollary 3.3.a: *Let $f \in \mathcal{M}^*(\mathbb{K})$. Then f is transcendental over $\mathbb{K}(x)$.*

Theorem 3.4: *Let $f \in \mathcal{M}(S)$. Then f belongs to $\mathcal{M}_b(S)$ if and only if $T(r, f)$ is bounded in $[0, R[$.*

Corollary 3.4.a: *Let $f \in \mathcal{M}_u(S)$. Then f is transcendental over $\mathcal{M}_b(S)$.*

Theorem 3.5: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$). Then for all $k \in \mathbb{N}^*$, we have $N(r, f^{(k)}) = N(r, f) + k\overline{N}(r, f)$ and $Z(r, f^{(k)}) \leq Z(r, f) + k\overline{N}(r, f) + O(1)$.*

Theorem 3.6: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$) and let $a_1, ..., a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \Big(\sum_{j=1, j \neq k}^{q} Z(r, f - a_j) \Big) + O(1).$$

Remark: The last Theorem does not hold in complex analysis. Indeed, let f be a meromorphic function in $\mathbb{C}$ omitting two values a and b, such as $f(x) = \dfrac{e^x}{e^x - 1}$. Then $Z(r, f - a) + Z(r, f - b) = 0$.

We can now state the Second Main Theorem.

Theorem 3.7: (Second Main Theorem) *Let* $\alpha_1, ..., \alpha_q \in \mathbb{K}$, *with* $q \geq 2$, *let* $Y = \{\alpha_1, ..., \alpha_q\}$ *and let* $f \in \mathcal{M}(\mathbb{K})$ *(resp.* $f \in \mathcal{M}_u(S)$*). Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r, f-\alpha_j) + \overline{N}(r,f) - Z_0^Y(r,f') - \log r + O(1) \quad \forall r \in I \quad (\text{resp. } \forall r \in J).$$

Moreover, if f *belongs to* $f \in \mathcal{A}(\mathbb{K})$ *(resp.* $f \in \mathcal{A}(S)$*), then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r, f-\alpha_j) - Z_0^Y(r,f') + O(1) \quad \forall r \in I \ (\text{resp. } \forall r \in J).$$

4. Nevanlinna Theory out of hole

Henceforth, we denote by L the interval $[R, +\infty[$. According to classical properties of analytic elements on infraconnected sets, is easy to have the following properties:

Lemma 4.1: *Let $f \in H(D)$ have no zero in D. Then $f(x)$ is of the form $\sum_{-\infty}^{q} a_n x^n$ with $|a_q|R^q > |a_n|R^n$ for all $n < q$.*

Definition: Let $f \in H(D)$ have no zero in D, $f(x) = \sum_{-\infty}^{q} a_n x^n$ with $|a_q|R^q > |a_n|R^n$ for all $n < q$ and $a_q = 1$. Then f will be called *a Motzkin factor associated to* S and the integer q will be called *the Motzkin index of f* and will be denoted by $m(f, S)$.

Theorem 4.2: *Let $f \in \mathcal{M}(D)$. We can write f in a unique way in the form $f^S f^0$ with $f^S \in H(D)$ a Motzkin factor associated to S and $f^0 \in \mathcal{M}(\mathbb{K})$, having no zero and no pole in S.*

A Nevanlinna Theory was made M. O. Hanyak and A. A. Kondratyuk in 2007 for functions meromorphic in the complex plane except at finitely many points where they can have an essential singularity.

In this part, we will give some relations between the characteristic function and Motzkin factors.

Given $f \in \mathcal{M}(D)$, for $r > R$. If $\alpha_1, ..., \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$, with respective multiplicity u_j, $1 \leq j \leq m$, then *the counting function of zeros* $Z_R(r, f)$ of f between R and r will denote by

$$Z_R(r, f) = \sum_{j=1}^{m} u_j(\log(r) - \log(|\alpha_j|)).$$

Similarly, if $\beta_1, ..., \beta_n$ are the distinct poles of f in $\Delta(0, R, r)$, with respective multiplicity v_j, $1 \leq j \leq m$, then *the counting function of poles* $N_R(r, f)$ of f between R and r is denoted by

$$N_R(r, f) = \sum_{j=1}^{n} v_j(\log(r) - \log(|\beta_j|)).$$

We put

$$T_R(r, f) = \max\big(Z_R(r, f), N_R(r, f)\big).$$

The counting function of zeros without counting multiplicity $\overline{Z}_R(r,f)$ is defined as:

$$\overline{Z}_R(r,f) = \sum_{j=1}^{m} \log(r) - \log(|\alpha_j|),$$

where $\alpha_1, ..., \alpha_m$ are the distinct zeros of f in $\Delta(0,R,r)$. Similarly, *the counting function of poles without counting multiplicity* $\overline{N}_R(r,f)$ is defined as:

$$\overline{N}_R(r,f) = \sum_{j=1}^{n} \log(r) - \log(|\beta_j|),$$

where $\beta_1, ..., \beta_n$ are the distinct poles of f in $\Delta(0,R,r)$.

Finally, putting $Y = \{a_1, ..., a_q\}$, we denote by $Z_R^Y(r,f')$ the counting function of zeros of f' on points x where $f(x) \notin Y$.

Theorem 4.3: *Let $f \in \mathcal{M}(D)$. Then, for all $r \in L$,* $\log(|f|(r)) - \log(|f|(R))$

$$= Z_R(r,f) - N_R(r,f) + m(f,S)(\log r - \log R).$$

Corollary 4.3.a *Let $f \in \mathcal{M}(D)$. Then $T_R(r, f)$ is identically zero if and only if f is a Motzkin factor.*

Let $f,\ g \in \mathcal{A}(D)$ satisfy $\log(|f|(r)) \leq \log(|g|(r))$ *for all $r \in L$. Then*

$$Z_R(r, f) \leq Z_R(r, g) + (m(g, S) - m(f, S))(\log(r) - \log(R)).$$

Theorem 4.4: *Let $f \in \mathcal{A}(D)$. Then, for $r \in L$,*

$$Z_R(r, f') \leq Z_R(r, f) - \log(r) + O(1).$$

We can now characterize the set $\mathcal{M}^*(D)$:

Theorem 4.5: *Let $f \in \mathcal{M}(D)$. The three following statements are equivalent:*

i) $\lim\limits_{r \to +\infty} \dfrac{T_R(r, f)}{\log(r)} = +\infty$ *for $r \in L$,*

ii) $\dfrac{T_R(r, f)}{\log(r)}$ *is unbounded,*

iii) f belongs to $\mathcal{M}^*(D)$.

Operations on $\mathcal{M}(D)$ work almost like for meromorphic functions in the whole field.

Theorem 4.6: *If $f,\ g \in \mathcal{M}(D)$. Then for every $b \in \mathbb{K}$ and $r \in L$, we have $T_R(r, f^n) = nT_R(r, f)$,*

$T_R(r, f.g) \leq T_R(r, f) + T_R(r, g) + O(\log(r))$,

$T_R(r, \frac{1}{f}) = T_R(r, f))$,

$T_R(r, f + g) \leq T_R(r, f) + T_R(r, g) + O(\log(r))$,

$T_R(r, f + b) = T_R(r, f) + O(\log(r))$,

$T_R(r, h \circ f) = T_R(r, f) + O(\log(r))$, *where h is a Moebius function.*

Moreover, if both f and g belong to $\mathcal{A}(D)$, then

$T_R(r, f+g) \leq \max(T_R(r, f), T_R(r, g)) + O(\log(r))$,

$T_R(r, fg) = T_R(r, f) + T_R(r, g)$.

Particularly, if $f \in \mathcal{A}^*(D)$, then

$T_R(r, f + b) = T_R(r, f) + O(1)$.

Given a polynomial $P(X) \in \mathbb{K}[X]$, then $T_R(r, P \circ f) = qT_R(r, f) + O(\log(r))$.

Theorem 4.7: *Every $f \in \mathcal{M}^*(D)$ is transcendental over $\mathcal{M}^w(D)$.*

Theorem 4.8: *Let $f \in \mathcal{M}(D)$. Then, for $r \in L$,*

$N_R(r, f^{(k)}) = N_R(r, f) + k\overline{N}_R(r, f)$ *and*

$Z_R(r, f^{(k)}) \leq Z_R(r, f) + k\overline{N}_R(r, f) + O(\log(r))$.

Like in the whole field, the Nevanlinna second Main Theorem is based on the following theorem:

Theorem 4.9: *Let $f \in \mathcal{M}(D)$ and let $a_1, ..., a_q \in$ $\mathbb{K}$ be distinct. Then*

$$(q-1)T_R(r,f) \leq \max_{1\leq k\leq q}\Big(\sum_{j=1, j\neq k}^{q} Z_R(r, f-a_j)\Big)+O(\log(r)).$$

We can now state and prove the Second Main Theorem for $\mathcal{M}(D)$.

Theorem 4.10: (Second Main Theorem) *Let $f \in \mathcal{M}(D)$, let $\alpha_1, ..., \alpha_q \in \mathbb{K}$, with $q \geq 2$ and let $Y = \{\alpha_1, ..., \alpha_q\}$. Then, for $r \in L$, $(q-1)T_R(r,f)$*

$$\leq \sum_{j=1}^{q} \overline{Z}_R(r, f-\alpha_j)+\overline{N}_R(r,f)-Z_R^Y(r,f')+O(\log(r)).$$

Particularly, if $f \in \mathcal{A}(D)$, then

$$(q-1)T_R(r,f) \leq \sum_{j=1}^{q} \overline{Z}_R(r, f-\alpha_j)-Z_R^Y(r,f')+O(\log(r)).$$

5. Immediate applications

We have the following immediate applications:

Theorem 5.1: *Let* $a_1,\ a_2 \in \mathbb{K}$ *(with* $a_1 \neq a_2$ *) and let* $f,\ g \in \mathcal{A}^*(\mathbb{K})$ *satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2$*). Then* $f = g$*.*

Theorem 5.2: *Let* $a_1,\ a_2,\ a_3 \in \mathbb{K}$ *(with* $a_i \neq a_j\ \forall i \neq j$*) and let* $f,\ g \in \mathcal{A}_u(d(0, R^-))$ *(resp.* $f,\ g \in \mathcal{A}^*(D)$*) satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2,\ 3$*). Then* $f = g$*.*

Theorem 5.3: *Let* $a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{K}$ *(with* $a_i \neq a_j\ \forall i \neq j$*) and let* $f, g \in \mathcal{M}^*(\mathbb{K})$ *satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2,\ 3,\ 4$*). Then* $f = g$*.*

Theorem 5.4: *Let* $a_1,\ a_2,\ a_3,\ a_4,\ a_5 \in \mathbb{K}$ *(with* $a_i \neq a_j\ \forall i \neq j$*) and let* $f, g \in \mathcal{M}_u(d(0, R^-))$ *(resp.* $f, g \in \mathcal{M}^*(D)$*) satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2,\ 3,\ 4,\ 5$*). Then* $f = g$*.*

Theorem 5.5:: *Let Λ be a non-degenerate elliptic curve of equation*
$y^2 = (x-a_1)(x-a_2)(x-a_3)$.

There do not exist $g,\ f \in \mathcal{M}(\mathbb{K})$ *such that* $g(t) = y,\ f(t) = x,\ t \in \mathbb{K}$.

There do not exist $g,\ f \in \mathcal{A}_u(d(0,R^-))$ *such that* $g(t) = y,\ f(t) = x,\ t \in d(0,R^-)$.

There do not exist $g,\ f \in \mathcal{A}^*(D)$ *such that* $g(t) = y,\ f(t) = x,\ t \in D$.

Theorem 5.6: *Let Λ be a curve of equation $y^q = P(x)$, $q \geq 2$, with $P \in \mathbb{K}[x]$ admitting n distinct zeros of order 1 with $n \geq 4$. There do not exist $g,\ f \in \mathcal{M}(\mathbb{K})$ such that $g(t) = y,\ f(t) = x,\ t \in \mathbb{K}$.*

Theorem 5.7: *Let Λ be a curve of equation $y^q = P(x)$, $q \geq 2$, with $P \in \mathbb{K}[x]$ admitting n distinct zeros of order 1 with $n \geq 5$. There do not exist $g,\ f \in \mathcal{M}_u(d(0,R^-))$ (resp. $g,\ f \in \mathcal{M}^*(D)$) such that $g(t) = y,\ f(t) = x,\ t \in d(0,R^-)$ (resp. $t \in D$).*

Another application concerns analytic functions:

Theorem 5.8: *Let $f,\ g \in \mathcal{M}(K)$ satisfy $g^m + f^n = 1$, with $\min(m,n) \geq 2$ and $\max(m,n) \geq 3$. Then f and g are constant.*

Theorem 5.9: *Let* $f,\ g \in \mathcal{M}(d(0, R^-))$ *(resp.* $f,\ g \in \mathcal{M}(D)$*) satisfy* $g^m + f^n = 1$*,with* $\min(m, n) \geq 3$ *and* $\max(m, n) \geq 4$*). Then* f *and* g *belong to* $\mathcal{M}_b(d(0, R^-))$ *(resp. to* $\mathcal{M}^w(D)$*). Moreover, if* $f,\ g \in \mathcal{A}(d(0, R^-)$ *(resp. if* $f,\ g \in \mathcal{A}(D)$*) satisfy* $g^m + f^n = 1$*, with* $\min(m, n) \geq 2$ *and* $(m, n) \neq (2, 2)$*, then* f *and* g *belong to* $\mathcal{A}_b(d(0, R^-)$*, (resp to* $\mathcal{A}^w(D)$*).*

6. Hayman's Conjecture:

Let us recall the famous Hayman conjecture on complex meromorphic functions:

Let f *be a meromorphic transcendental function in* $\mathbb{C}$*. Then, for every* $n \in \mathbb{N}^*$*,* $f^n f'$ *takes every value* $b \in \mathbb{C}$ *infinitely many times.*

That conjecture was easily proven by W. Hayman himself for every $n \geq 3$, then by E. Mues for $n = 2$ and at last by W. Bergweiler for $n = 1$.

The same problem appeared in $\mathcal{M}^*(\mathbb{K})$, in $\mathcal{M}_u(S)$ and in $\mathcal{M}^*((D)$.The Nevanlinna theory lets us obtain a solution in each cases whenever $n \geq 3$. The solution was given by Jacqueline Ojeda.

Theorem 6.1: *Let $f \in \mathcal{M}^*(\mathbb{K})$, (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$),. Then for every $n \geq 3$, $f^n f'$ takes every value $b \in \mathbb{K}$ infinitely many times.*

The big difficulties begin with $n = 2$ and $n = 1$. Theorem 6.2 was proven in 2013 by J. Ojeda and the first author:

Teorem 6.2: *Let $f \in \mathcal{M}^*(\mathbb{K})$. Then $f^2 f'$ takes every value $b \in \mathbb{K}$ infinitely many times.*

Concerning the case $n = 1$, by Bézivin's Theorem (Theorem 2.4), we have this.

Theorem 6.3: *Let $f \in \mathcal{M}(\mathbb{K})$, $r > 0$ and let $\xi(f, r)$ be the number of poles of f in $d(0, r)$. If in $[1, +\infty[$, $\xi(f, r)$ admits an upper bound of the form $\xi(f, r) \leq r^q$, ($q > 0$) then $f'f$ takes every value $b \in \mathbb{K}$ infinitely many times.*

7. Small functions

Definitions: For each $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(S)$, resp. $f \in \mathcal{M}(D)$), we will denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(S)$, $\mathcal{M}_f(D)$) the set of functions $h \in \mathcal{M}(\mathbb{K})$ (resp. $h \in \mathcal{M}(S)$, $h \in \mathcal{M}(D)$) such that $T_R(r,h) = o(T_R(r,f))$, $r \in I$, (resp. $r \in J$, resp. $r \in L$). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(S)$, resp. $f \in \mathcal{A}(D)$) we will denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(S)$, resp. $\mathcal{A}_f(D)$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$,(resp. $\mathcal{M}_f(S) \cap \mathcal{A}(S)$, resp.$\mathcal{M}_f(D) \cap \mathcal{A}(D)$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(S)$, resp. $\mathcal{M}_f(D)$) are called *small meromorphic functions with respect to f, small functions* in brief. Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(S)$, resp. $f \in \mathcal{A}(D)$) these functions are called *small analytic functions with respect to f, small functions in brief.*

A small function w with respect to a function $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$, $f \in \mathcal{M}(D)$) will be called *a quasi-exceptional small function for f* if $f - w$ has finitely many zeros in D.

Theorem 7.1: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$). Then f admits at most one quasi-exceptional small function. Moreover, if f has finitely many poles, then f admits no quasi-exceptional small function.*

Corollary 7.1.a: *Let $f \in \mathcal{A}^*(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(S)$, resp. $f \in \mathcal{A}^*(D)$). Then f has no quasi-exceptional small function.*

By applying the Second Main Theorem for meromorphic functions outside a hole and following step by step the classical previous p-adic works, we are able to obtain uniqueness theorems, which will not be listed in this paper.

First, we will show a Second Main Theorem for Three Small Functions for meromorphic functions outside a hole. It holds as well as in complex analysis, where it was showed first. Notice that this theorem was generalized to any finite set of small functions by K. Yamanoi in complex analysis, through methods that have no equivalent on a p-adic field.

Theorem 7.2: *Let $f \in \mathcal{M}^*(\mathbb{K})$, (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$ and let $w_1,\ w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $\in \mathcal{M}_f(d(0, R^-))$, resp. $\in \mathcal{M}_f(D)$) be pairwise distinct. Then:*

$$T(r,f) \le \sum_{j=1}^{3} \overline{Z}(r, f - w_j) + o(T(r,f))$$

(resp. $T(r,f) \le \sum_{j=1}^{3} \overline{Z}(r, f - w_j) + o(T(r,f))$, *resp.* $T_R(r,f) \le \sum_{j=1}^{3} \overline{Z}_R(r, f - w_j) + o(T_R(r,f))$*).*

Corollary 7.2.a: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$ resp. $f \in \mathcal{M}^*(D)$) and let $w_1,\ w_2 \in \mathcal{A}_f(\mathbb{K})$ (resp. $w_1,\ w_2 \in \mathcal{A}_f(S)$ resp. $w_1,\ w_2 \in \mathcal{A}_f(D)$) be distinct. Then*

$$T(r,f) \le \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r,f) + o(T(r,f)).$$

(resp.

$$T(r,f) \le \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r,f) + o(T_R(r,f)).$$

resp. $T_R(r,f)$

$$\le \overline{Z}_R(r, f - w_1) + \overline{Z}_R(r, f - w_2) + \overline{N}_R(r,f) + o(T_R(r,f))).$$

Corollary 7.2.b: *Let $f \in \mathcal{A}^*(D)$ and let $w_1,\ w_2 \in \mathcal{A}_f(D)$ be distinct. Then*

$$T_R(r,f) \leq \overline{Z}_R(r,f-w_1)+\overline{Z}_R(r,f-w_2)+o(T_R(r,f)).$$

The other problem that we can solve with the help of Theorem 7.1 concerns branched functions.

Definitions: Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-))$, resp. $f \in \mathcal{M}^*(D)$) and let $w \in \mathcal{M}_f(\mathbb{K})$ (resp. $w \in \mathcal{M}_f(d(0,R^-))$, resp. $w \in \mathcal{M}_f(D)$). Then w is called *a perfectly branched function with respect to f* if all zeros of $f-w$ are multiple except maybe finitely many and w is called *a totally branched function with respect to f* if all zeros of $f-w$ are multiple, without exception. Particularly, the definition applies to constants.

Theorem 7.3: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-))$, resp. $f \in \mathcal{M}^*(D)$). Then f admits at most four perfectly branched values.*

Theorem 7.4: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}^*(D)$) having finitely many poles. Then f admits at most one perfectly branched rational function.*

Corollary 7.4.a: *Let* $f \in \mathcal{A}^*(\mathbb{K})$ *(resp.* $f \in \mathcal{A}^*(D)$*). Then* f *admits at most one perfectly branched rational function.*

Theorem 7.5: *Let* $f \in \mathcal{M}_u(d(0, R^-))$*, having finitely many poles. Then* f *admits at most two perfectly branched rational functions.*

Corollary 7.5.a: *Let* $f \in \mathcal{A}_u(d(0, R^-))$*. Then* f *admits at most two perfectly branched rational functions.*

8. New applications of the Nevanlinna Theory

Definitions: Recall that two functions f and g meromorphic in a set B are said *to share a set* $X \subset \mathbb{K}$*, counting multiplicity*, or *C.M.* in brief, if for each $b \in X$, when $f(x) - b$ has a zero of order q at a point $a \in B$, then there exists $c \in X$ such that $g(x) - c$ also has a zero of order q at a. And the functions f and g are said *to share* X *ignoring multiplicity* or *I.M.* in brief, if $f^{-1}(X) = g^{-1}(X)$.

By Theorem 5.3, two meromorphic functions in $\mathbb{K}$ sharing 4 points I.M. are identical , by Theorem 5.4 two meromorphic functions in S or in D sharing 5 points I.M. are identical, by Theorem 5.1, two entire functions sharing 2 points I.M. are identical and by Theorem 5.2 two meromorphic functions in S or in D sharing 3 points I.M. are identical. Here we will first examine two meromorphic functions sharing a few points C.M.

Theorem 8.1: *Let* $f,\ g \in \mathcal{M}_u(S)$ *(resp. let* $f,\ g \in \mathcal{M}^*(D)$*) share C.M.* 4 *points* $a_j \in \mathbb{K} \cup \{\infty\},\ j = 1, 2, 3, 4$. *Then* $f \equiv g$.

Theorem 8.2: *Let* $f,\ g \in \mathcal{M}_u(S)$ *(resp. let* $f,\ g \in \mathcal{M}^*(D)$*) have finitely many poles and share C.M.* 3 *points* $a_j \in \mathbb{K} \cup \{\infty\},\ j = 1, 2, 3$. *Then* $f \equiv g$.

Corollary 8.2.a: *Let* $f,\ g \in \mathcal{A}_u(S)$ *(resp.* $f,\ g \in \mathcal{A}^*(D)$*) share C.M.* 3 *points* $a_j \in \mathbb{K},\quad j = 1, 2, 3$. *Then* $f \equiv g$.

Theorem 8.3 is not immediate and has a similar version in complex analysis for meromorphic functions provided with a finite growth order that is not integral. Here, we don't need any hypothesis on the growth order.

Theorem 8.3: *Let $f,\ g \in \mathcal{M}^*(\mathbb{K})$ share C.M. 3 points $a_j \in \mathbb{K} \cup \{\infty\},\ j = 1, 2, 3$. Then $f \equiv g$.*

In the particular case of functions $f,\ g \in \mathcal{M}_u(S)$ or functions $f,\ g \in \mathcal{M}^*(D))$ having finitely many poles and sharing poles C.M., we can add this theorem:

Theorem 8.4: *Let $f,\ g \in \mathcal{M}_u(S)$, (resp. let $f,\ g \in \mathcal{M}^*(D))$ have finitely many poles in S (resp. in D) and share C.M. two values a, b and poles. Then $f \equiv g$.*

Our main theorems are Theorems 8.5 and 8.6 that follow the same kind of reasoning as in the classical case. We denote by $Y_1 = \{\alpha_1, ..., \alpha_k\}$ and $Y_2 = \{\beta_1, \beta_2\}$ the two sets satisfying

$$(\mathcal{H}) \quad \Big(\prod_{j=1}^{k}(\beta_1 - \alpha_j)\Big)^2 \neq \Big(\prod_{j=1}^{k}(\beta_2 - \alpha_j)\Big)^2.$$

Theorem 8.5: *Let $f,\ g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{M}_u(S)$, resp. let $f,\ g \in \mathcal{M}^*(D)$) have finitely many poles in $\mathbb{K}$ (resp. in S, resp. in D) and share Y_1 C.M. and Y_2 I.M. Then $f \equiv g$.*

Corollary 8.5.a: *Let $f,\ g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{M}_u(S)$, resp. let $f,\ g \in \mathcal{M}^*(D)$) have finitely many poles in $\mathbb{K}$ (resp. in S, resp in D) and share a value α C.M. and Y_2 I.M. If $(\alpha-\beta_1)^2 \neq (\alpha-\beta_2)^2$, then $f \equiv g$.*

Corollary 8.5.b: *Let $f,\ g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{A}_u(S)$, resp. let $f,\ g \in \mathcal{A}^*(D)$) and share Y_1 C.M. and Y_2 I.M. Then $f \equiv g$.*

Corollary 8.5.c: *Let $f,\ g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{A}_u(S)$, resp. let $f,\ g \in \mathcal{A}^*(D)$) and share a value α C.M. and Y_2 I.M. If $(\alpha - \beta_1)^2 \neq (\alpha - \beta_2)^2$, then $f \equiv g$.*

Theorem 8.6: *Let $f,\ g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{M}_u(S)$, resp. let $f,\ g \in \mathcal{M}^*(D)$) have finitely many poles in $\mathbb{K}$ (resp. in S, resp. in D) and share Y_1 I.M. and Y_2 C.M. Then $f \equiv g$.*

Corollary 8.6.a: *Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in $\mathbb{K}$ (resp. in S, resp. in D) and share a value α I.M. and Y_2 C.M. If $(\alpha-\beta_1)^2 \neq (\alpha-\beta_2)^2$, then $f \equiv g$.*

Corollary 8.6.b: *Let $f, g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(S)$, resp. let $f, g \in \mathcal{A}^*(D)$) and share Y_1 I.M. and Y_2 C.M. Then $f \equiv g$.*

Corollary 8.6.c: *Let $f, g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(S)$, resp. let $f, g \in \mathcal{A}^*(D)$) and share a value α I.M. and Y_2 C.M. If $(\alpha - \beta_1)^2 \neq (\alpha - \beta_2)^2$, then $f \equiv g$.*

It is known that if two functions $f, g \in \mathcal{A}(\mathbb{K})$ share separately two values $a, b \in \mathbb{K}$ C.M., then $f \equiv g$. However, here the hypothesis *f, g share Y_1 and share Y_2* cannot be compared: for example, concerning Y_2, f and g are not supposed to share β_1 or β_2 separately. The same remark applies to meromorphic functions having finitely many poles.

Results recently presented by J.F. Chen show the interest of complex functions of the form $f(x)f(x+b)$. Similar studies were made in a p-adic field by Liu Gang and Meng Chao.

Here we will generalize that kind of study on the field $\mathbb{K}$. it is proven that if two complex entire functions f and g are such that $f(x)^n f(x+c)$ and $g(x)^n g(x+c)$ share 1 C.M. with $n \geq 6$, then either fg is a constant t_1 such that $t_1^{n+1} = 1$, or $\frac{f}{g}$ is a constant t_2 such that $t_2^{n+1} = 1$. Here, on the field $\mathbb{K}$, we can obtain better results.

On the other hand, we can find similar results of uniqueness by Vu Hoai An, Pham Ngoc Hoa and Ha Hui Khoai, for meromorphic functions on a p-adic field involving derivatives, sharing 1 C.M. or I.M., also involving derivatives. Here we will examine functions of the form $f(x)^n(f(x+c))^m$, $g(x)^n(g(x+c))^m$ sharing a rational function and we will look for branched values and quasi-exceptional values of such functions.

Notation: We denote by $\mathbb{N}^*$ the set of strictly positive integers. On $\mathbb{Z}$, we denote by $| \ . \ |_\infty$ the Archimedean absolute value.

Theorem 8.7: *Let $a \in C(0,1)$, let $b \in \mathbb{K}$ and let $f,\ g \in \mathcal{M}^*(\mathbb{K})$ have finitely many poles and take $m,\ n\ \mathbb{N}^*$ with $m \neq n$. If $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share C.M. a rational function $Q \in \mathbb{K}(x)$, $Q \not\equiv 0$ and if $n+m \geq 5$, then $\dfrac{f}{g}$ is a constant t such that $t^{n+m} = 1$. Moreover, if $f,\ g \in \mathcal{A}^*(\mathbb{K})$, if $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share C.M. a constant $l \neq 0$ and if $n+m \geq 4$, then $\dfrac{f}{g}$ is a constant t such that $t^{n+m} = 1$.*

Theorem 8.8: *Let $a \in C(0,1)$ and let $b \in S$ (resp. let $b \in \mathbb{K}$) and let $f,\ g \in \mathcal{M}_u(S)$ (resp. let $f,\ g \in \mathcal{M}^*(D)$) have finitely many poles in S (resp. in D) and take $n,\ m \in \mathbb{N}^*$ with $n \neq m$. If $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share C.M. a function $\tau \in \mathcal{M}(S)$ (resp. a function $\tau \in \mathcal{M}(D)$) having finitely many zeros and poles in S (resp. in D), then $\dfrac{f}{g}$ is a constant t such that $t^{n+m} = 1$.*

Next, in the paper by J.F.Chen, it was shown that given a complex entire function f and $b \in \mathbb{C} \setminus \{0\}$, a function of the form $f^n(x)f(x+b) - c$ (with $c \neq 0$) has infinitely many zeros in $\mathbb{C}$ provided $n \geq 3$. On the field $\mathbb{K}$, such a result is trivial since an entire functions and even a meromorphic function with finitely many poles (which is not a rational function) takes every value infinitely many times. But we can ask the question regarding in general functions $f \in \mathcal{M}^*(\mathbb{K})$, $f \in \mathcal{M}_u(S)$, $f \in \mathcal{M}^*(D)$.

Theorem 8.9: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$), let $a \in C(0,1)$, let $b \in \mathbb{K}$ (resp. $b \in S$, resp. $b \in \mathbb{K}$) and let $w \in \mathcal{M}(\mathbb{K})$ (resp. $w \in \mathcal{M}(S)$, resp. $w \in \mathcal{M}(D)$) be a non identically zero small function with respect to f. If $n,\ m \in \mathbb{N}$ are such that $|n-m|_\infty \geq 5$, then $f^n(x)f^m(ax+b)-w$ has infinitely many zeros in $\mathbb{K}$ (resp. in S, resp. in D).*

Corollary 8.9.a: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$), let $a \in C(0,1)$, let $b \in \mathbb{K}$ (resp. $b \in S$, resp. $b \in \mathbb{K}$). If $n,\ m \in \mathbb{N}$ are such that $|n-m|_\infty \geq 5$, then $f^n(x)f^m(ax+b)$ takes every nonzero value infinitely many times in $\mathbb{K}$ (resp. in S, resp. in D).*

Remarks: 1) Of course, the hypothesis $w \neq 0$ must not be excluded. Indeed, let $h \in \mathcal{A}^*(\mathrm{I\!K})$ and let $f(x) = \dfrac{1}{h(x)}$. Then a function of the form $f^n(x)f(ax+b)$ has no zero in $\mathrm{I\!K}$.

2) On the other hand, it is known and easily seen that if $a = 1$ and $b = 0$, a function $f^n - w$ has infinitely many zeros for every $n \geq 3$ and that f^2 takes every nonzero value c infinitely many times because given a square root l of c, then $f^2 - c = (f - l)(f + l)$ and at least one of the two values l and $-l$ is taken infinitely many times. We can ask whether a meromorphic function of the form $f^2(x) - w$ always has infinitely many zeros when w is not a constant.

3) Concerning Theorem 8.9, it is easily proven that if f is a meromorphic function with finitely many poles and w a small function, then $f - w$ has infinitely many zeros. So it is useless here to add a corollary concerning $f^n(x)f^m(ax + b) - w$ when f has finitely many poles.

Theorem 8.10: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$), let $a \in C(0,1)$, let $b \in \mathbb{K}$ (resp. $b \in S$, resp. $b \in \mathbb{K}$) and let n, $m \in \mathbb{N}^*$. If $3|n-m|_\infty > 2(n+m)+4$, then $f^n(x)f^m(ax+b)$ does not admit 4 distinct perfectly branched values and if $3|n-m|_\infty \geq 2(n+m+1)$, then $f^n(x)f^m(ax+b)$ does not admit 4 distinct totally branched values. Moreover, if $4|n-m|_\infty > 3(n+m)+4$, then $f^n(x)f^m(ax+b)$ does not admit 3 distinct perfectly branched values and if $4|n-m|_\infty \geq 3(n+m)+4$, then $f^n(x)f^m(ax+b)$ does not admit 3 distinct totally branched values.*