A Heat Equation on the Ring of Adèles

Samuel Estala–Arias, based on joint works with Ph. D. Manuel López-Crúz and M.C. Victor Aguilar-Arteaga

Sixth International Conference on p-adic Mathematical Physics and its Applications

CINVESTAV, Mexico City, October 24, 2017

Outline

Outline

Outline

Outline

Outline

Outline

Outline

1. Riemann, (1854) found that geometry and Physics are interconnected at a fundamental level and describes Riemannian geometry as model for space.
2. Riemann commented that this kind of hypothesis should be, or have to be, consistent with Physics.
3. In very small distances, time-space might not be a manifold. In the last decades of the last century many other models emegered to explain what happen at very small distance.

1. Riemann, (1854) found that geometry and Physics are interconnected at a fundamental level and describes Riemannian geometry as model for space.
2. Riemann commented that this kind of hypothesis should be, or have to be, consistent with Physics.
3. In very small distances, time-space might not be a manifold. In the last decades of the last century many other models emegered to explain what happen at very small distance.

1. Riemann, (1854) found that geometry and Physics are interconnected at a fundamental level and describes Riemannian geometry as model for space.
2. Riemann commented that this kind of hypothesis should be, or have to be, consistent with Physics.
3. In very small distances, time-space might not be a manifold. In the last decades of the last century many other models emegered to explain what happen at very small distance.

1. Riemann, (1854) found that geometry and Physics are interconnected at a fundamental level and describes Riemannian geometry as model for space.
2. Riemann commented that this kind of hypothesis should be, or have to be, consistent with Physics.
3. In very small distances, time-space might not be a manifold. In the last decades of the last century many other models emegered to explain what happen at very small distance.

1. In 80's, I. Volovich, began to use p-adic numbers instead of the real numbers to explore models of space-time at the Plank scales. Space-time is now considered non–Archimedean.
2. In 1989, Yu. I Manin propose to use or incorporate all p-adic fields and the real line to construct a model of space-time.

1. In 80's, I. Volovich, began to use p-adic numbers instead of the real numbers to explore models of space-time at the Plank scales. Space-time is now considered non–Archimedean.
2. In 1989, Yu. I Manin propose to use or incorporate all p-adic fields and the real line to construct a model of space-time.

1. In 80's, I. Volovich, began to use p-adic numbers instead of the real numbers to explore models of space-time at the Plank scales. Space-time is now considered non–Archimedean.
2. In 1989, Yu. I Manin propose to use or incorporate all p-adic fields and the real line to construct a model of space-time.

Outline

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers and let $\mathbb{P}$ be the set of prime numbers. For any $p \in \mathbb{P}$, the p-completion of the integers, $\mathbb{Z}$, is given by

$$\mathbb{Z}_p \cong \left\{ \sum_{i=0}^{\infty} x_i p^i \right\}$$

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers and let $\mathbb{P}$ be the set of prime numbers. For any $p \in \mathbb{P}$, the p-completion of the integers, $\mathbb{Z}$, is given by

$$\mathbb{Z}_p \cong \left\{ \sum_{i=0}^{\infty} x_i p^i \right\}$$

Another equivalent way to define the p-adic completion of the integers is given as follows:

$$\begin{aligned}\mathbb{Z}_p &\cong \varprojlim_{m\in\mathbb{N}\cup\{0\}} \mathbb{Z}/p^m\mathbb{Z} \\ &\cong \left\{x \in \prod \mathbb{Z}/p^m\mathbb{Z},\ x_i = f(x_j),\ i \leq j\right\}\end{aligned}$$

Another equivalent way to define the p-adic completion of the integers is given as follows:

$$\mathbb{Z}_p \cong \varprojlim_{m \in \mathbb{N} \cup \{0\}} \mathbb{Z}/p^m\mathbb{Z}$$
$$\cong \left\{ x \in \prod \mathbb{Z}/p^m\mathbb{Z},\ x_i = f(x_j),\ i \leq j \right\}$$

For any prime number p, the **field of p–adic numbers** $\mathbb{Q}_p$ is the field of fractions of the topological ring $\mathbb{Z}_p$, namely,

$$\mathbb{Q}_p = \mathrm{Quot}(\mathbb{Z}_p).$$

For any prime number p, the field of p–adic numbers $\mathbb{Q}_p$ is the field of fractions of the topological ring $\mathbb{Z}_p$, namely,

$$\mathbb{Q}_p = \mathrm{Quot}(\mathbb{Z}_p).$$

1. The p–adic order on $\mathbb{Z}_p$ extends to an order on $\mathbb{Q}_p$ and this produces a non–Archimedean valuation on $\mathbb{Q}_p$ which makes it a second countable and totally disconnected locally compact topological field.
2. The unit ball on $\mathbb{Q}_p$ corresponds to the maximal compact and open subring $\mathbb{Z}_p$. The Haar measure dx_p on the additive group $\mathbb{Q}_p$ is normalized to be a probability measure on $\mathbb{Z}_p$.

1. The p–adic order on $\mathbb{Z}_p$ extends to an order on $\mathbb{Q}_p$ and this produces a non–Archimedean valuation on $\mathbb{Q}_p$ which makes it a second countable and totally disconnected locally compact topological field.
2. The unit ball on $\mathbb{Q}_p$ corresponds to the maximal compact and open subring $\mathbb{Z}_p$. The Haar measure dx_p on the additive group $\mathbb{Q}_p$ is normalized to be a probability measure on $\mathbb{Z}_p$.

1. The p–adic order on $\mathbb{Z}_p$ extends to an order on $\mathbb{Q}_p$ and this produces a non–Archimedean valuation on $\mathbb{Q}_p$ which makes it a second countable and totally disconnected locally compact topological field.
2. The unit ball on $\mathbb{Q}_p$ corresponds to the maximal compact and open subring $\mathbb{Z}_p$. The Haar measure dx_p on the additive group $\mathbb{Q}_p$ is normalized to be a probability measure on $\mathbb{Z}_p$.

The finite adèle ring $\mathbb{A}_f$ of the rational numbers $\mathbb{Q}$ is the *restricted direct product* of the fields $\mathbb{Q}_p$ with respect to the subrings $\mathbb{Z}_p$, viz.,

$$\mathbb{A}_f = \left\{ (x_p)_{p\in\mathbb{P}} \in \prod_{p\in\mathbb{P}} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for almost any } p \in \mathbb{P} \right\}.$$

The **finite adèle ring** $\mathbb{A}_f$ of the rational numbers $\mathbb{Q}$ is the *restricted direct product* of the fields $\mathbb{Q}_p$ with respect to the subrings $\mathbb{Z}_p$, viz.,

$$\mathbb{A}_f = \left\{ (x_p)_{p\in\mathbb{P}} \in \prod_{p\in\mathbb{P}} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for almost any } p \in \mathbb{P} \right\}.$$

Let $S \subset \mathbb{P}$ be a finite set of prime numbers. The space of *S–adèles* of $\mathbb{Q}$ is the product ring

$$\mathbb{A}_S = \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

Let $S \subset \mathbb{P}$ be a finite set of prime numbers. The space of S–*adèles* of $\mathbb{Q}$ is the product ring

$$\mathbb{A}_S = \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

The restricted direct product topology on the ring $\mathbb{A}_f$ is the topology of the inductive limit

$$\mathbb{A}_f = \varinjlim_{\substack{S \subset \mathbb{P} \\ |S| < \infty}} \mathbb{A}_S,$$

$\mathbb{A}_f$, is a second countable and totally disconnected locally compact topological ring and contains $\mathbb{Q}$ as a dense subset. The profinite completion of the integers

$$\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$$

is the maximal, compact and open subring of $\mathbb{A}_f$. The Haar measure $d\mu = \prod_{p \in \mathbb{P}} dx_p$ on $\mathbb{A}_f$ is a probability measure on $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$.

$\mathbb{A}_f$, is a second countable and totally disconnected locally compact topological ring and contains $\mathbb{Q}$ as a dense subset. The profinite completion of the integers

$$\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$$

is the maximal, compact and open subring of $\mathbb{A}_f$. The Haar measure $d\mu = \prod_{p\in\mathbb{P}} dx_p$ on $\mathbb{A}_f$ is a probability measure on $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$.

$\mathbb{A}_f$, is a second countable and totally disconnected locally compact topological ring and contains $\mathbb{Q}$ as a dense subset. The profinite completion of the integers

$$\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$$

is the maximal, compact and open subring of $\mathbb{A}_f$. The Haar measure $d\mu = \prod_{p \in \mathbb{P}} dx_p$ on $\mathbb{A}_f$ is a probability measure on $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$.

Outline

The Archimedean property

Given two real numbers x, y, with $x > 0$, there exists a natural number n, such that $nx > y$.

The Archimedean property appears to be an "axiom" in the construction of the real numbers, only after a complete system of axioms is given or considered.

The Archimedean property

Given two real numbers x, y, with $x > 0$, there exists a natural number n, such that $nx > y$.

The Archimedean property appears to be an "axiom" in the construction of the real numbers, only after a complete system of axioms is given or considered.

Let us begin again with the rational numbers, $\mathbb{Q}$, and the concept of ultrametric distance.

Definition

An (additive invariant) ultrametric seminorm, $|\cdot|$, on $\mathbb{Q}$ is a real valued function that satisfies the following properties

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$.

(iv) The distance function $d(x, y) = |x - y|$ is invariant under additive traslations.

Let us begin again with the rational numbers, $\mathbb{Q}$, and the concept of ultrametric distance.

Definition

An (additive invariant) ultrametric seminorm, $|\cdot|$, on $\mathbb{Q}$ is a real valued function that satisfies the following properties

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$.

(iv) The distance function $d(x, y) = |x - y|$ is invariant under additive traslations.

Let us begin again with the rational numbers, $\mathbb{Q}$, and the concept of ultrametric distance.

Definition

An (additive invariant) ultrametric seminorm, $|\cdot|$, on $\mathbb{Q}$ is a real valued function that satisfies the following properties

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$.

(iv) The distance function $d(x, y) = |x - y|$ is invariant under additive traslations.

Let us begin again with the rational numbers, $\mathbb{Q}$, and the concept of ultrametric distance.

Definition

An (additive invariant) ultrametric seminorm, $|\cdot|$, on $\mathbb{Q}$ is a real valued function that satisfies the following properties

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$.

(iv) The distance function $d(x, y) = |x - y|$ is invariant under additive traslations.

Let us begin again with the rational numbers, $\mathbb{Q}$, and the concept of ultrametric distance.

Definition

An (additive invariant) ultrametric seminorm, $|\cdot|$, on $\mathbb{Q}$ is a real valued function that satisfies the following properties

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$.

(iv) The distance function $d(x, y) = |x - y|$ is invariant under additive traslations.

Let us begin again with the rational numbers, $\mathbb{Q}$, and the concept of ultrametric distance.

Definition

An (additive invariant) ultrametric seminorm, $|\cdot|$, on $\mathbb{Q}$ is a real valued function that satisfies the following properties

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$.

(iv) The distance function $d(x, y) = |x - y|$ is invariant under additive traslations.

How many (regular) additive invariant ultrametrics does $\mathbb{Q}$ have?

Theorem (Ostrowski)

Every non trivial norm on $\mathbb{Q}$ is equivalent to the usual absolute value, or to a p-adic norm, $|\cdot|_p$, for some prime number p.

How many (regular) additive invariant ultrametrics does $\mathbb{Q}$ have?

Theorem (Ostrowski)

Every non trivial norm on $\mathbb{Q}$ is equivalent to the usual absolute value, or to a p-adic norm, $|\cdot|_p$, for some prime number p.

Schedule

Let $\psi(n)$ denote the second Chebyshev function defined by the relation

$$e^{\psi(n)} = lcm(1, 2, \ldots, n) \qquad (n \in \mathbb{N}).$$

Denote by $\Lambda(n)$ the von Mangoldt function given by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(n)$ denote the second Chebyshev function defined by the relation

$$e^{\psi(n)} = lcm(1, 2, \dots, n) \qquad (n \in \mathbb{N}).$$

Denote by $\Lambda(n)$ the von Mangoldt function given by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For any integer number n, define the second symmetric Chebyshev function by

$$\psi(n) = \begin{cases} \frac{n}{|n|}\psi(|n|) & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases}$$

and the symmetric von Mangoldt function by (extending) the relation

$$e^{\Lambda(n)} = \frac{e^{\psi(n)}}{e^{\psi(n-1)}}.$$

For any integer number n, define the second symmetric Chebyshev function by

$$\psi(n) = \begin{cases} \frac{n}{|n|}\psi(|n|) & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases}$$

and the symmetric von Mangoldt function by (extending) the relation

$$e^{\Lambda(n)} = \frac{e^{\psi(n)}}{e^{\psi(n-1)}}.$$

For each integer number n, denote by $e^{\psi(n)}\mathbb{Z} \subset \mathbb{Q}$ the family of additive subgroups of $\mathbb{Q}$. If n is positive, $e^{\psi(n)}\mathbb{Z}$ is an ideal of $\mathbb{Z}$ and, if n is negative, $e^{\psi(n)}\mathbb{Z}$ is a fractional ideal of $\mathbb{Q}$.

For each integer number n, denote by $e^{\psi(n)}\mathbb{Z} \subset \mathbb{Q}$ the family of additive subgroups of $\mathbb{Q}$. If n is positive, $e^{\psi(n)}\mathbb{Z}$ is an ideal of $\mathbb{Z}$ and, if n is negative, $e^{\psi(n)}\mathbb{Z}$ is a fractional ideal of $\mathbb{Q}$.

The filtration $(m < 0 < n)$

$$\{0\} \subset \cdots \subset e^{\psi(n)}\mathbb{Z} \subset \cdots \subset \mathbb{Z} \subset \cdots \subset e^{\psi(m)}\mathbb{Z} \subset \cdots \subset \mathbb{Q}$$

has the properties

$$\bigcap_{n\in\mathbb{Z}} e^{\psi(n)}\mathbb{Z} = \{0\} \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} e^{\psi(n)}\mathbb{Z} = \mathbb{Q}.$$

The filtration $(m < 0 < n)$

$$\{0\} \subset \cdots \subset e^{\psi(n)}\mathbb{Z} \subset \cdots \subset \mathbb{Z} \subset \cdots \subset e^{\psi(m)}\mathbb{Z} \subset \cdots \subset \mathbb{Q}$$

has the properties

$$\bigcap_{n\in\mathbb{Z}} e^{\psi(n)}\mathbb{Z} = \{0\} \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} e^{\psi(n)}\mathbb{Z} = \mathbb{Q}.$$

The collection $\{e^{\psi(n)}\mathbb{Z}\}_{n\in\mathbb{Z}}$ is a neighbourhood base of zero for an additive invariant topology on $\mathbb{Q}$. This topology is called here the finite adelic topology of $\mathbb{Q}$.

The collection $\{e^{\psi(n)}\mathbb{Z}\}_{n\in\mathbb{Z}}$ is a neighbourhood base of zero for an additive invariant topology on $\mathbb{Q}$. This topology is called here the finite adelic topology of $\mathbb{Q}$.

For any element $x \in \mathbb{Q}$ define the order of x as:

$$ord(x) := \begin{cases} \max\{n : x \in e^{\psi(n)}\mathbb{Z}\} & \text{if } n \neq 0, \\ \infty & \text{if } n = 0. \end{cases}$$

With this order, define a nonnegative function

$$d : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

given by

$$d(x, y) = e^{\psi(-ord(x-y))}.$$

With this order, define a nonnegative function

$$d : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

given by

$$d(x, y) = e^{\psi(-ord(x-y))}.$$

The completion of $\mathbb{Q}$, denoted by $\overline{\mathbb{Q}}$, is the quotient ring formed by Cauchy sequences modulo trivial Cauchy sequences. Any element of $\overline{\mathbb{Q}}$ finite can be written uniquely as

$$x = \sum_{n=ord(x)}^{\infty} x(n) \cdot e^{\psi(n)},$$

where $x(n) = 0, 1, \ldots, e^{\Lambda(n+1)} - 1$.

The completion of $\mathbb{Q}$, denoted by $\overline{\mathbb{Q}}$, is the quotient ring formed by Cauchy sequences modulo trivial Cauchy sequences. Any element of $\overline{\mathbb{Q}}$ finite can be written uniquely as

$$x = \sum_{n=ord(x)}^{\infty} x(n) \cdot e^{\psi(n)},$$

where $x(n) = 0, 1, \ldots, e^{\Lambda(n+1)} - 1$.

There is an isomorphism of topological rings

$$\mathbb{A}_f \cong \mathbb{N}^{-1}\widehat{\mathbb{Z}} \cong \overline{\mathbb{Q}},$$

which preserves the inclusion of $\mathbb{Q}$ on both rings.

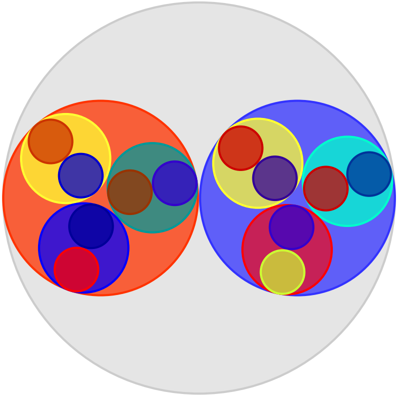

Figure: The decomposition of $\mathbb{A}_f$

Theorem

If d is any regular non–Archimedean metric on $\mathbb{A}_f$, then d is determined by an ordered pair $(\alpha(n), \beta(n))$ of sequences of natural numbers totally ordered by divisibility and cofinal with $\mathbb{N}$.

[CE2016] Cruz–López, Manuel, Estala–Arias, Samuel. *Additive invariant ultrametrics on the finite adèle group of $\mathbb{Q}$.* P–Adic Numbers Ultrametric Analysis and Applications 04/2016; 8(2):89–114.

Theorem

If d is any regular non–Archimedean metric on $\mathbb{A}_f$, then d is determined by an ordered pair $(\alpha(n), \beta(n))$ of sequences of natural numbers totally ordered by divisibility and cofinal with $\mathbb{N}$.

[CE2016] Cruz–López, Manuel, Estala–Arias, Samuel. *Additive invariant ultrametrics on the finite adèle group of $\mathbb{Q}$*. P–Adic Numbers Ultrametric Analysis and Applications 04/2016; 8(2):89–114.

The ultrametric constructed using the second Chebyshev function gives some interesting integrals. For $\sigma = \Re(s) > 0$,

$$\int_{\widehat{\mathbb{Z}}} \|x\|^{s-1}\, d\mu(x) = \sum_{n=1}^{\infty} e^{-(s-1)\psi(n)} e^{-\psi(n)} \left(1 - e^{-\Lambda(n+1)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1 - e^{-\Lambda(n+1)}}{e^{s\psi(n)}} < \int_0^1 x^{\sigma-1}\, dx.$$

The ultrametric constructed using the second Chebyshev function gives some interesting integrals. For $\sigma = \Re(s) > 0$,

$$\int_{\widehat{\mathbb{Z}}} \|x\|^{s-1} \, d\mu(x) = \sum_{n=1}^{\infty} e^{-(s-1)\psi(n)} e^{-\psi(n)} (1 - e^{-\Lambda(n+1)})$$
$$= \sum_{n=1}^{\infty} \frac{1 - e^{-\Lambda(n+1)}}{e^{s\psi(n)}} < \int_0^1 x^{\sigma-1} dx.$$

Outline

For any α, consider the operator

$$D^\alpha : Dom(A) \subset L^2(\mathbb{A}_f) \to L^2(\mathbb{A}_f)$$

defined by the following diagram :

$$\begin{array}{ccc} L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \\ {\scriptstyle D^\alpha}\downarrow & & \downarrow{\scriptstyle f \mapsto \|\cdot\|^\alpha f} \\ L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \end{array} \tag{1}$$

For any α, consider the operator

$$D^\alpha : Dom(A) \subset L^2(\mathbb{A}_f) \to L^2(\mathbb{A}_f)$$

defined by the following diagram :

$$\begin{array}{ccc} L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \\ {\scriptstyle D^\alpha}\downarrow & & \downarrow{\scriptstyle f \mapsto \|\cdot\|^\alpha f} \\ L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \end{array} \qquad (1)$$

A pseudodifferential equation of the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D^{\alpha}u(x,t) = 0, \ x \in \mathbb{A}_f, \ t > 0 \\ \lim_{t\to 0} u(x,t) = f(x) \end{cases} \tag{2}$$

for some appropriate function $f \in L^2(\mathbb{A}_f)$, is a finite adelic counterpart of the Archimedean homogeneous heat equation.

The Hille-Yosida theorem implies a diagram:

$$\begin{array}{ccc} L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \\ {\scriptstyle S(t)}\downarrow & & \downarrow{\scriptstyle f \mapsto f \exp(-t\|\cdot\|^\alpha)} \\ L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \end{array}$$

In order to find an explicit expression for $S(t)$ it is necessary to introduce the heat kernel:

$$Z(x,t) = \int_{\mathbb{A}_f} \chi(-x\xi) \exp(-t \left\|\xi\right\|^{\alpha}) d\xi.$$

$$Z(x,t) = \sum_{\substack{n\in\mathbb{Z} \\ e^{\psi(n)}\le\|x\|^{-1}}} e^{\psi(n)} \left\{ \exp(-te^{\alpha\psi(n)}) - \exp(-te^{\alpha\psi(n+1)}) \right\}.$$

Theorem

Let $\alpha > 0$ and let $S(t)$ be the C_0–semigroup generated by the operator $-D^\alpha$. The solution of the abstract Cauchy problem is given by $u(x,t) = Z(x,t) * f(x)$, for $t \geq 0$ and $f \in Dom(D^\alpha)$.

Outline

Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ be the complete ring of Adèles.

Being a finite pruduct of locally compact topological rings, we have

1. $\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{R})$
2. $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$
3. $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}_f} \otimes \mathcal{F}_{\mathbb{R}}$
4. $D^{\alpha,\beta} = D^{\alpha} \otimes D^{\beta}$
5. $S_{\mathbb{A}}(t) = S_{\mathbb{A}_f}(t) \otimes S_{\mathbb{R}}(t)$

Being a finite pruduct of locally compact topological rings, we have

1. $\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{R})$
2. $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$
3. $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}_f} \otimes \mathcal{F}_{\mathbb{R}}$
4. $D^{\alpha,\beta} = D^{\alpha} \otimes D^{\beta}$
5. $\mathcal{S}_{\mathbb{A}}(t) = \mathcal{S}_{\mathbb{A}_f}(t) \otimes \mathcal{S}_{\mathbb{R}}(t)$

Being a finite pruduct of locally compact topological rings, we have

1. $\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{R})$
2. $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$
3. $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}_f} \otimes \mathcal{F}_{\mathbb{R}}$
4. $D^{\alpha,\beta} = D^{\alpha} \otimes D^{\beta}$
5. $S_{\mathbb{A}}(t) = S_{\mathbb{A}_f}(t) \otimes S_{\mathbb{R}}(t)$

Being a finite pruduct of locally compact topological rings, we have

1. $\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{R})$
2. $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$
3. $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}_f} \otimes \mathcal{F}_{\mathbb{R}}$
4. $D^{\alpha,\beta} = D^{\alpha} \otimes D^{\beta}$
5. $\mathcal{S}_{\mathbb{A}}(t) = \mathcal{S}_{\mathbb{A}_f}(t) \otimes \mathcal{S}_{\mathbb{R}}(t)$

Being a finite pruduct of locally compact topological rings, we have

1. $\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{R})$
2. $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$
3. $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}_f} \otimes \mathcal{F}_{\mathbb{R}}$
4. $D^{\alpha,\beta} = D^{\alpha} \otimes D^{\beta}$
5. $S_{\mathbb{A}}(t) = S_{\mathbb{A}_f}(t) \otimes S_{\mathbb{R}}(t)$

Being a finite pruduct of locally compact topological rings, we have

1. $\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{R})$
2. $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$
3. $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}_f} \otimes \mathcal{F}_{\mathbb{R}}$
4. $D^{\alpha,\beta} = D^{\alpha} \otimes D^{\beta}$
5. $\mathcal{S}_{\mathbb{A}}(t) = \mathcal{S}_{\mathbb{A}_f}(t) \otimes \mathcal{S}_{\mathbb{R}}(t)$

Outline

- Cruz–López, Manuel, Aguilar–Arteaga, Victor A., Estala–Arias, Samuel. *A Heat Equation on some Adic Completions of* $\mathbb{Q}$ *and Ultrametric Analysis*. P–Adic Numbers Ultrametric Analysis and Applications 07/2017; 9(3):165–182
- Cruz–López, Manuel, Estala–Arias, Samuel. *Additive invariant ultrametrics on the finite adèle group of* $\mathbb{Q}$. P–Adic Numbers Ultrametric Analysis and Applications 04/2016; 8(2):89–114.
- Cruz–López, Manuel, Estala–Arias, Samuel. *Fourier Analysis on the ring of Adèles*. Publicación en progreso (2017).
- Aguilar–Arteaga, Victor A., Estala–Arias, Samuel. *A heat equation on the ring of Adèles*. Publicación en progreso (2017).

Gracias por su atención