An Ultrametric Route towards Berry-Keating

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This talk is based on an ongoing collaboration with
Arghya Chattopadhyay, Parikshit Dutta
and Suvankar Dutta.

Caution: There are loose ends!

Outline

Introduction & Motivation

Phase Space Description of the Unitary Matrix Model

Unitary Matrix Model for the Symmetric Zeta-function

UMM for the Local ζ-function and Attempts at a Synthesis

Zeta function: infinite sum and product

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- extended to real $s > 1$ by Chebyshev.
- analytically continued to the complex s-plane as a meromorphic function by Riemann.

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The symmetric function $\xi(s) = \frac{1}{2}\pi^{-s/2} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is an entire function that satisfies $\xi(s) = \xi(1-s)$. Its zeroes are at the non-trivial zeroes of ζ, at $s = \gamma_m = \frac{1}{2} + it_m$.

Zeroes as the spectrum

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Rudnick-Sarnak extended it to higher correlators, also to zeroes of

Dirichlet L-functions: $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p\in\text{primes}} \frac{\chi(p)}{(1-p^{-s})}$.

Search for the Hamiltonian

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$$\begin{array}{ccc} \text{Riemann zeroes} & & \text{Energy eigenvalues} \\ \updownarrow & \sim & \updownarrow \\ \text{Primes} & & \text{Periods} \\ & & \text{(of primitive periodic orbits)} \end{array}$$

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  Restrict the values of x and p.

Outline

Conformal map

The eigenvalues of large $N \times N$ unitary matrices gives a density $\rho(\theta) = \sum \delta(\theta - \theta_i)$ (distribution function) on the unit circle.

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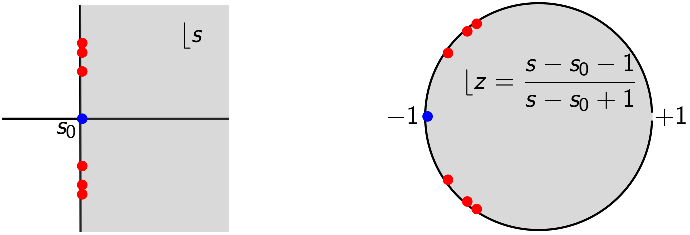
The eigenvalues of large $N \times N$ unitary matrices gives a density $\rho(\theta) = \sum \delta(\theta - \theta_i)$ (distribution function) on the unit circle. Given a distribution on the line $\mathrm{Re}\ s = s_0$, one can find a Gaussian Unitary Ensemble (GUE) such that its eigenvalue distribution is related to it.



$$s - s_0 = \frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} = i \cot\frac{\theta}{2}$$

One-plaquette UMM

The partition function of the one-plaquette model is defined by:

$$\mathcal{Z} = \int \mathcal{D}U \exp\left[-N \sum_{n=0}^{\infty} \frac{\beta_n}{n} \left(\text{Tr } U^n + \text{Tr } U^{\dagger n}\right)\right] = \int \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \, e^{-N^2 S_{\text{eff}}(\theta_i)}$$

where, $S_{\text{eff}}(\theta_i) = \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{2\beta_n}{n} \cos(n\theta_i) + \frac{1}{2} \sum_{i \neq j} \ln\left(4 \sin^2 \frac{\theta_i - \theta_j}{2}\right)$

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In the large N limit, $x = \frac{i}{N} \in [0,1]$ and $\theta_i \to \theta(x)$

$$S[\theta] = \sum_{n=1}^{\infty}\int_0^1 dx\, \frac{2\beta_n}{n}\cos n\theta(x) + \frac{1}{2}\int_0^1 dx \fint_0^1 dy\, \ln\left(4\sin^2\frac{\theta(x)-\theta(y)}{2}\right)$$

Saddle point

The saddle point of the action, determined by

$$\fint \frac{d\theta'}{2\pi}\rho(\theta')\cos\left(\frac{\theta-\theta'}{2}\right) = \sum_{n=1}^{\infty} 2\beta_n \sin n\theta \quad \left(= \frac{dV(\theta)}{d\theta}\right)$$

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$$\rho(\theta) = 2\mathrm{Re}\left[R(e^{i\theta})\right] - 1 = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left[R\Big((1+\epsilon)e^{i\theta}\Big) - R\Big((1-\epsilon)e^{i\theta}\Big)\right]$$

UMM in terms of Irreps (Schematic)

The PF of a UMM can be expanded in terms of the irreducible representions (irreps) of U(N)

$$\mathcal{Z} \sim \sum_{R \in \text{irreps}} \sum_{\vec{k},\vec{\ell}} \alpha(\vec{\beta},\vec{k})\alpha(\vec{\beta},\vec{\ell})\,\chi_R(C(\vec{k}))\chi_R(C(\vec{\ell}))$$

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(where $\chi_R(C(\vec{k}))$ is the character of the conjugacy class $C(\vec{k})$ of the permutation group $S_{K=\sum nk_n}$.). The following have been used

$$\prod_n (\text{Tr } U^n)^{k_n} = \sum_R \chi_R(C(\vec{k}))\text{Tr}_R(U)$$

$$\int \mathcal{D}U\, \text{Tr}_R(U)\text{Tr}_{R'}(U^\dagger) = \delta_{RR'}$$

Young diagrams and momenta

Irreps can be labelled by the number of boxes in Young diagrams.
In the large N limit

$$\mathcal{Z} = \int \mathcal{D}h(x) \int d\vec{k}\, d\vec{\ell} \exp\left(-N^2 S_{\rm eff}[h(x), \vec{k}, \vec{\ell}]\right)$$

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The variable h are the conjugate momenta of the eigenvalues θ.

Partition function in the phase space

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There is a density $\Omega(\theta, h)$ in the phase space, such that

$$\int dh\, \Omega(\theta, h) = \rho(\theta) \quad \text{and} \quad \int d\theta\, \Omega(\theta, h) = u(h)$$

Phase space description can lead to a Hamiltonian.

Outline

One plaqutte UMM and the Li numbers

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- This determines the parameters of the one plaquette model:

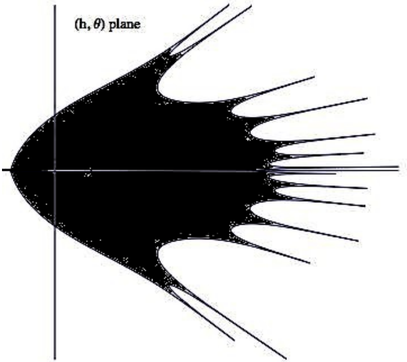
$$\beta_n = -\frac{1}{2n \ln 2}\lambda_n = \frac{1}{2 \ln 2} \oint_{\mathcal{C}_1} \frac{ds}{2\pi i} \frac{s^{n-1}}{(s-1)^n + 1} \ln \xi(s)$$

in terms of the Li numbers

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} s^{n-1} \ln \xi(s) \Big|_{s=1} = \sum_i \left[1 - \left(1 - \frac{1}{\gamma_i} \right)^n \right]$$

Phase space density of the UMM of $\xi(s)$

The density in the phase space is



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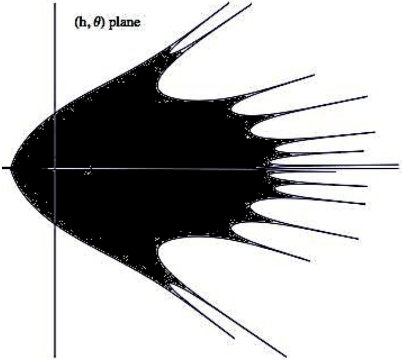

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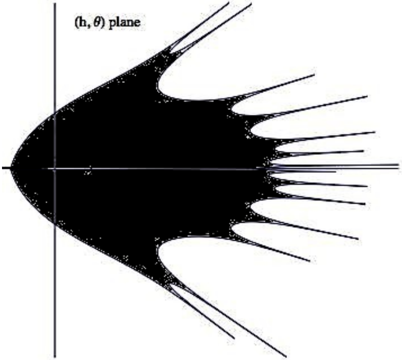

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It turns out that $h(x) \sim \widetilde{J}(x)$, the fluctuating part of the counting function.

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Local zeta and the resolvent

The local zeta function at the prime p, $\zeta_p(s) = (1 - p^{-s})^{-1}$ does not have any zero, but has equally spaced simple poles at $s = \frac{2\pi i}{\ln p} n$ $(n \in \mathbb{Z})$ on the vertical line $\mathrm{Re}(s) = 0$.

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These poles can be brought on the unit circle on $z = \frac{s-1}{s+1}$ plane.

$$R_<(z) = 1 + \frac{z}{(1-z)^2} \frac{p^{-s(z)}}{1 - p^{-s(z)}}, \qquad R_>(z) = -\frac{z}{(1-z)^2} \frac{p^{s(z)}}{1 - p^{s(z)}}$$

The resolvent above satisfies all the properties ($R_<(0) = 1$, $R_>(z \to \infty) = 0$ and $R_<(z) + R_>(1/z) = 1$). (Caveat)

A well-known measure

As everyone knows

$$\int_{\mathbb{Z}_p^\times} |h|_p^{s-1}\, dh = \frac{(1-p^{-1})p^{-s}}{(1-p^{-s})}, \qquad \mathbb{Z}_p^\times = \left\{ h \in \mathbb{Q}_p : \, |h|_p < 1 \right\}$$

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So $2R_<(z) - 1 = p \int_{\mathbb{Z}_p^\times} dh \left(1 + \frac{2z}{(p-1)(1-z)^2}\, |h|_p^{\frac{1+z}{1-z}-1} \right)$

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This is suggestive of a phase space density

$$\Omega_p(\theta, h) = p - \frac{p}{2(p-1)\sin^2\left(\frac{\theta}{2}\right)} \, |h|_p^{-i\cot\left(\frac{\theta}{2}\right)-1} \sim p - \frac{p^{-in\cot\left(\frac{\theta}{2}\right)}}{2(p-1)\sin^2\left(\frac{\theta}{2}\right)}$$

Vladimirov derivative and Kozyrev wavelets

$p^{-n\alpha}$ is the eigenvalue of the generalized Vladimirov derivative $D^{\alpha}_{(p)}$ for any complex number α:

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Fluctuating part $\sim \displaystyle\int_0^\infty d\xi \, \xi^{-i\cot\frac{\theta}{2}} \frac{dJ_p(\xi)}{d\xi}$.

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The Hilbert space $\mathcal{H}_{(p)}$ of the quantum Hamiltonian is expected to be spanned by the Kozyrev wavelets, which are eigenfunctions of the Vladimirov derivative.

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$$\int_0^{\infty} d\xi \, \xi^{-i \cot \frac{\theta}{2}} \frac{dJ_p}{d\xi} \to \int_0^{\infty} d\xi \, \xi^{-i \cot \frac{\theta}{2}} \ln p \frac{dJ_p}{d\xi}$$

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Combining all p

$$\beta_m = \sum_p \ln p \, \beta_m^{(p)} \sim \int d\left(\cot\frac{\theta}{2}\right) e^{-im\theta} \sum_{n=1}^{\infty} \left\langle \Psi_n | \mathbb{D}^{-i\cot\frac{\theta}{2}} | \Psi_n \right\rangle$$

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$$\sum_p \ln p \, \frac{dJ_p(\xi)}{d\xi} = \frac{d\psi(\xi)}{d\xi} = 1 - \underbrace{\sum_i \xi^{\gamma_i - 1}}_{\text{non-trivial zeroes}} - \sum_n \xi^{2n-1}$$

Divergence

Keeping only the non-trivial zeroes γ_i

$$\beta_m \sim \int d\xi\, \xi^{-i\cot\frac{\theta}{2}+\gamma_i-1} = \int d(\ln\xi)\, e^{\mathrm{Re}(\gamma_i)\ln\xi + i\left(\mathrm{Im}(\gamma_i)-\cot\frac{\theta}{2}\right)\ln\xi}$$

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Clearly μ has to be independent of i. The reflection symmetry of ζ-function implies that $\mu > 1$ and if Riemann hypothesis is true $\mu > \frac{1}{2}$.

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Leads to a one-parameter family of Hamiltonians

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Similar approach to the Dirichlet L-functions, and indeed other more general L-functions, may be worth the effort.

In summary, we attempt to get to the elusive Hamiltonian for the zeta-function by starting at the local zeta-function at the p-th place. This suggests a phase space picture with the hint of a Hamiltonian. We attempt to combine this for all primes.

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¡Gracias! Thank you!