Pseudodifferential operators and parabolic type equations on adeles

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Consider the *ring of finite adeles* over $\mathbb{Q}$ denoted $\mathbb{A}_f$ and defined by

$$\mathbb{A}_f = \big\{(x_2, x_3, x_5, \ldots) : x_p \in \mathbb{Q}_p, \text{ and } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\big\},$$

and the *ring of adeles of* $\mathbb{Q}$ denoted $\mathbb{A}$ and defined by

$$\mathbb{A} = \big\{(x_\infty, x_2, x_3, x_5 \ldots) : x_p \in \mathbb{Q}_p, \text{ and } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\big\},$$

where $\mathbb{Q}_p$ are the fields of p-adic numbers, $\mathbb{Z}_p$ are the ideals of p-adic integers. Alternatively, we can define $\mathbb{A}_f$ and $\mathbb{A}$ as *the restricted products* of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. The componentwise addition and multiplication give to $\mathbb{A}_f$ and $\mathbb{A}$ ring structures. Furthermore, $\mathbb{A}_f$ (correspondingly, $\mathbb{A}$) can be made into a locally compact topological ring by taking as a base for the *restricted product topology* all the sets of the form $U \times \prod_{p \notin S} \mathbb{Z}_p$ where S is any finite set of primes (correspondingly, containing ∞), and U is any open subset in $\prod_{p \in S} \mathbb{Q}_p$.

In the talk we show that the function

$$\|x\| := \begin{cases} \max_p \frac{|x_p|_p}{p} & \text{if } x \in \prod_p \mathbb{Z}_p, \\ \max_p |x_p|_p & \text{if } x \notin \prod_p \mathbb{Z}_p \end{cases}$$

defined for arbitrary $x \in \mathbb{A}_f$ induces the non-Archimedean metric $\rho_f(x, y) := \|x - y\|$ for the ring of finite adeles. It is shown that the ring of adeles equipped with this metric becomes a complete metric space, the topology induced by the presented metric coincides with the restricted product topology and the Fourier transform of a radial function with respect to this metric is again a radial function. Moreover, balls and spheres in this metric are compact sets and their volumes are related with the second Chebyshev function.

Also we show that the metric on adeles may be introduced as

$$\rho_{\mathbb{A}}(x, y) := |x_\infty - y_\infty|_\infty + \|x_f - y_f\|,$$

and that this metric induces the restricted product topology.

Based on the presented metrics adelic analogues of Taibleson pseudodifferential operators and Lizorkin spaces of the second kind are introduced, their properties are discussed. For parabolic type equations involving these pseudodifferential operators the heat kernels are constructed and it is shown that a general solution of a Cauchy problem can be obtained as a convolution of the initial data with the heat kernel.

The talk is based on a joint work with W. A. Zúñiga-Galindo [1].

References

[1] S. Torba and W. Zúñiga-Galindo, *Parabolic Type Equations and Markov Stochastic Processes on Adeles*, to appear in Journal of Fourier Analysis and Applications, available at arXiv:1206.5213.